MONASH University

MONASH BUSINESS SCHOOL

# ETC3550/ETC5550 Applied forecasting

## Ch9. ARIMA models

OTexts.org/fpp3/

# Point forecasts

1. Rearrange ARIMA equation so $y_t$ is on LHS.
2. Rewrite equation by replacing $t$ by $T + h$.
3. On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with $h = 1$. Repeat for $h = 2, 3, \ldots$.

# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

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- $v_{T+1|T} = \hat{\sigma}^2$ for all ARIMA models regardless of parameters and orders.
- Multi-step prediction intervals for ARIMA(0,0,$q$):

$$y_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2\right], \qquad \text{for } h = 2, 3, \ldots.$$

# Prediction intervals

- Prediction intervals **increase in size with forecast horizon**.
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed**.
- Prediction intervals tend to be too narrow.
  - the uncertainty in the parameter estimates has not been accounted for.
  - the ARIMA model assumes historical patterns will not change during the forecast period.
  - the ARIMA model assumes uncorrelated future errors

# Seasonal ARIMA models

| ARIMA | $\underbrace{(p, d, q)}$ | $\underbrace{(P, D, Q)_m}$ |
|---|---|---|
| | $\uparrow$ | $\uparrow$ |
| | Non-seasonal part of the model | Seasonal part of of the model |

where $m$ = number of observations per year.

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- $(1-\phi_1 B)$: Non-seasonal AR(1)
- $(1-\Phi_1 B^4)$: Seasonal AR(1)
- $(1-B)$: Non-seasonal difference
- $(1-B^4)$: Seasonal difference
- $(1+\theta_1 B)$: Non-seasonal MA(1)
- $(1+\Theta_1 B^4)$: Seasonal MA(1)

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All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t &= (1+\phi_1)y_{t-1} - \phi_1 y_{t-2} + (1+\Phi_1)y_{t-4} \\ &\quad - (1+\phi_1+\Phi_1+\phi_1\Phi_1)y_{t-5} + (\phi_1+\phi_1\Phi_1)y_{t-6} \\ &\quad - \Phi_1 y_{t-8} + (\Phi_1+\phi_1\Phi_1)y_{t-9} - \phi_1\Phi_1 y_{t-10} \\ &\quad + \varepsilon_t + \theta_1\varepsilon_{t-1} + \Theta_1\varepsilon_{t-4} + \theta_1\Theta_1\varepsilon_{t-5}. \end{aligned}$$

# Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

**ARIMA(0,0,0)(0,0,1)$_{12}$ will show:**

- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ….

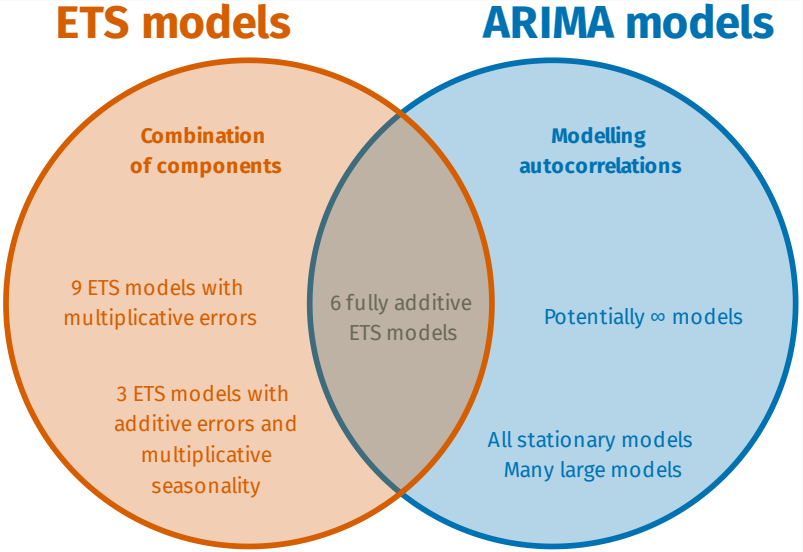
**ARIMA(0,0,0)(1,0,0)$_{12}$ will show:**

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

# ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

# ARIMA vs ETS


ETS models
ARIMA models
Combination of components
Modelling autocorrelations
9 ETS models with multiplicative errors
6 fully additive ETS models
Potentially ∞ models
3 ETS models with additive errors and multiplicative seasonality
All stationary models
Many large models

# Equivalences

| ETS model | ARIMA model | Parameters |
|---|---|---|
| ETS(A,N,N) | ARIMA(0,1,1) | $\theta_1 = \alpha - 1$ |
| ETS(A,A,N) | ARIMA(0,2,2) | $\theta_1 = \alpha + \beta - 2$ |
| | | $\theta_2 = 1 - \alpha$ |
| ETS(A,$A_d$,N) | ARIMA(1,1,2) | $\phi_1 = \phi$ |
| | | $\theta_1 = \alpha + \phi\beta - 1 - \phi$ |
| | | $\theta_2 = (1 - \alpha)\phi$ |
| ETS(A,N,A) | ARIMA(0,0,$m$)(0,1,0)$_m$ | |
| ETS(A,A,A) | ARIMA(0,1,$m$ + 1)(0,1,0)$_m$ | |
| ETS(A,$A_d$,A) | ARIMA(1,0,$m$ + 1)(0,1,0)$_m$ | |