10/2/2023

Poisson CLT and Poisson Approximation

<u>Example:</u> $X_1, X_2, \ldots, X_n$ iid Poisson(λ)

$$f_{X_n}(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 1, 2, \ldots$$

$$E[X_n] = \lambda \quad , \quad V(X_n) = \lambda$$

$$M_{X_n}(t) = e^{\lambda(e^t - 1)}$$

Recall:

$$M_{X_n}(t) = E[e^{tX_n}]$$
$$= E[g(X_n)]$$

We want to look at $\bar{X}_n$ in the limit

$$\bar{X}_n \xrightarrow{P} \lambda \qquad \text{LLN}$$

We also consider

$$Z_n = \frac{\bar{X}_n - \lambda}{\sqrt{\frac{\lambda}{n}}} = \frac{\sqrt{n}\,(\bar{X}_n - \lambda)}{\sqrt{\lambda}}$$

Consider $M_{Z_n}(t)$ as $n \to \infty$

$$M_{Z_n}(t) = E\left[e^{tZ_n}\right] = E\left[e^{t\frac{\sqrt{n}(\bar{X}_n - \lambda)}{\sqrt{\lambda}}}\right]$$

$$= E\left[\underbrace{e^{-\sqrt{n\lambda}\,t}}_{\text{constant}} e^{\sqrt{\frac{n}{\lambda}}\bar{X}_n t}\right]$$

$$= e^{-\sqrt{n\lambda}\,t} E\left[e^{\sqrt{\frac{n}{\lambda}}\bar{X}_n t}\right]$$

$$= e^{-\sqrt{n\lambda}\,t} M_{\sqrt{\frac{n}{\lambda}}\bar{X}_n}(t) \leftarrow$$

$$M_{\bar{X}_n}(t) = \left[M_{\frac{X}{n}}(t) \right]^n = \left[M_X\left(\frac{t}{n}\right) \right]^n$$

Poisson.

$$= \left[e^{\lambda(e^{\frac{t}{n}} - 1)} \right]^n$$

$$= e^{n\lambda(e^{\frac{t}{n}} - 1)}$$

see ch. 4 Sec 5

Prop C

Prop D

Thus

$$M_{Z_n}(t) = e^{-\sqrt{n\lambda}\, t} \, M_{\sqrt{\frac{n}{\lambda}}\,\bar{X}_n}(t)$$

$$= e^{-\sqrt{n\lambda}\, t} \, M_{\bar{X}_n}\left(\sqrt{\frac{n}{\lambda}}\, t\right)$$

$$= e^{-\sqrt{n\lambda}\, t} \, e^{n\lambda\left(e^{\frac{t}{\sqrt{n\lambda}}} - 1\right)}$$

Now send $n \to \infty$

take $\log_e$ first

$$\log M_{Z_n}(t) = -\sqrt{n\lambda}\, t + n\lambda\left(e^{\frac{t}{\sqrt{n\lambda}}} - 1\right)$$

expand the exponential part

recall $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$\sqrt{n\lambda}\cdot t$

$$\log M_{Z_n}(t) = -\sqrt{n\lambda}\,t + n\lambda\left[\cancel{1} + \frac{t}{\sqrt{n\lambda}} + \frac{1}{2}\frac{t^2}{n\lambda} + \frac{1}{3!}\frac{t^3}{(n\lambda)^{3/2}} + \cdots \cancel{-1}\right]$$

$$= \frac{1}{2} t^2 + o\left(\frac{1}{\sqrt{n}}\right)$$

going to zero.

$\therefore$ with $n \to \infty$ we have

$$\log M_{Z_n}(t) \to \frac{1}{2} t^2$$

$$M_{Z_n}(t) \to e^{\frac{1}{2}t^2}$$ $=$ the MGF of $N(0,1)$

Thus $Z_n = \dfrac{\sqrt{n}(\bar{X}_n - \lambda)}{\sqrt{\lambda}} \xrightarrow{d} Z \sim N(0,1)$ $\square$

by continuity thm.

MGF see Section 4.5.

Prop A. X has a CDF $F_X(x)$, $f_X(x)$

$M_X(t)$ uniquely identifies $F_X(x)$

Prop C $M_X(t)$ $Y = a + bX$

$M_Y(t) = e^{at} M_X(bt)$

Prop D $Z = X + Y$ indep $M_Z(t) = M_X(t) M_Y(t)$

In the book Example A p. 181

$$X_n \sim \text{Poisson}(\lambda).$$

$$Z_n = \frac{X_n - E[X_n]}{\sqrt{V(X_n)}}$$ ✓

X_n last value in a sample.

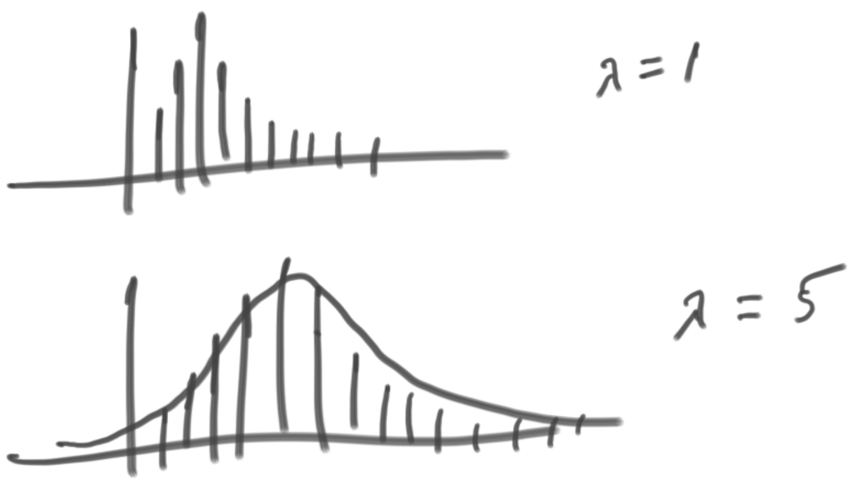
Find the limiting distribution of Z_n as $n \to \infty$

$E[X_n] = \lambda_n$

$\lambda_n \to \infty$

This is the limiting distribution of a Poisson r.v. X_n

see Figure 2.6 p. 43



Problem 3

Normal Approximation to the Poisson.

In the book Example F p. 187

$$X_n \sim Bin(n, p)$$

$$E[X_n] = np$$

De Moivre - Laplace Limit Thm

$$P\left(\frac{X_n - np}{\sqrt{np(1-p)}} \le x \right) \longrightarrow N(0,1)$$

as $n \to \infty$

Normal Approximation of the Binomial.

Central Limit Thm

Let $X_1, \ldots, X_n$ iid r.v.'s each having mean μ and variance σ^2.

Then the distribution of

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$$

where $S_n = \sum_{i=1}^{n} X_i$ tends to the standard normal.

$$\lim_{n\to\infty} P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x \right) = \Phi(x)$$

$-\infty < x < \infty$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt.$

see Thm B p. 184

<u>Pf</u>: Let $\mu = 0$

Assume the $M_X(t)$ exists.

...

Thm: Let $F_1, F_2, \ldots$ be a sequence of CDFs and the corresponding PDFs be $f_1, f_2, \ldots$

If $\lim_{n \to \infty} f_n(x) = f(x) \ \forall x$, $f(x)$ a PDF, then F_n converges in distribution to the CDF F corresponding to f.

See T table

<u>Example</u>: $X_n \sim t_n$.

prob. 27

PDF $$f_{X_n}(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \cdot \left(1+\frac{x^2}{n}\right)^{-\frac{n-1}{2}} \frac{1}{\sqrt{n}}$$

$$-\infty < x < \infty.$$

Aside: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

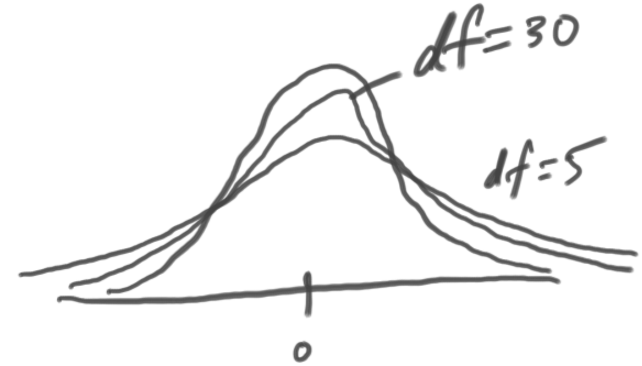
$$= \frac{1}{\sqrt{2\pi}} \boxed{\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{\frac{n}{2}}}} \left(1+\frac{x^2}{n}\right)^{-\frac{n}{2}} \left(1+\frac{x^2}{n}\right)^{-\frac{1}{2}}$$

as $n \to \infty$

$$f_{X_n}(x) \to \frac{1}{\sqrt{2\pi}} \cdot 1 \cdot e^{-\frac{x^2}{2}} \cdot 1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

PDF $N(0,1)$

Note: $$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{\frac{n}{2}}} \longrightarrow 1$$



stirling formula

$$\Gamma(n+1) = n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

try it with some numbers.

Prediction $\qquad E[Y|X].$

homework 5

3 N Poisson.

Normal Approximation

$$z_\lambda = \frac{N - \lambda}{\sqrt{\lambda}}$$

Exercise 2. CLT Normal

CLT Exp.

prog 6.r

$$\bar{X}_n \,\dot\sim\, N\left(\mu, \frac{\sigma^2}{n}\right) \qquad SD(\bar{X}) = \frac{\sigma}{\sqrt{n}}.$$

X matrix

$$\begin{array}{c|cccc|c} & 1 & 2 & \cdots & 100 & \\ \begin{matrix}1\\2\\\vdots\\30\end{matrix} & [\;\;] & [\;\;] & \cdots & [\;\;] & \\ & \bar{X}_1 & \bar{X}_2 & & \bar{X}_{100} & \to \text{hist} \end{array}$$

Exercise1

see example5.r   Week 06

Derived Distributions   Ch. 6

χ^2 , t , F χ^2 . Beta.

$\sim N$

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

$\sqrt{\chi^2}$ /   indep

χ^2

$$F = \frac{SSB/df}{SSW/df}$$

χ^2 /

indep