10/2/2023

Poisson CLT and Poisson Approximation

<u>Example</u>: $X_1, X_2, \dots, X_n$ iid Poisson (λ)

$$f_{X_n}(x) = \frac{e^{-\lambda} \lambda^x}{x!} \qquad x = 1, 2, \dots$$

$$E[X_n] = \lambda \qquad V(X_n) = \lambda$$

$$M_{X_n}(t) = e^{\lambda(e^t - 1)}$$

Recall: $M_{X_n}(t) = E[e^{tX_n}]$
$= E[g(X)]$

We want to look at $\bar{X}_n$ in the limit

$$\bar{X}_n \xrightarrow{p} \mu \qquad \text{LLN}$$

We also consider

$$Z_n = \frac{\bar{X}_n - \lambda}{\sqrt{\frac{\lambda}{n}}} = \frac{\sqrt{n}\,(\bar{X}_n - \lambda)}{\sqrt{\lambda}} \qquad \text{CLT}$$

Consider $M_{Z_n}(t)$ as $n \to \infty$.

$$M_{Z_n}(t) = E\left[e^{tZ_n}\right] = E\left[e^{t\frac{\sqrt{n}(\bar{X}_n - \lambda)}{\sqrt{\lambda}}}\right]$$

$$= E\Big[\underbrace{e^{-\sqrt{n\lambda}\cdot t}}_{\text{constant}}\; e^{\sqrt{\frac{n}{\lambda}}\,\bar{X}_n t}\Big]$$

$$= e^{-\sqrt{n\lambda}\, t}\; E\left[e^{\sqrt{\frac{n}{\lambda}}\bar{X}_n t}\right].$$

$$= e^{-\sqrt{n\lambda}\, t}\; M_{\sqrt{\frac{n}{\lambda}}\cdot \bar{X}_n}(t)$$

constant.

$$M_{\bar{X}_n}(t) = \left[M_{\frac{X}{n}}(t) \right]^n = \left[M_X\left(\frac{t}{n}\right) \right]^n$$

Poisson.

$$= \left[e^{\lambda\left(e^{\frac{t}{n}} - 1\right)} \right]^n$$

$$= e^{n\lambda\left(e^{\frac{t}{n}} - 1\right)}$$

see Ch. 4

Prop C

Prop D

Thus

$$M_{Z_n}(t) = e^{-\sqrt{n\lambda}\, t} \, M_{\sqrt{\frac{n}{\lambda}}\,\overline{X}_n}(t)$$

$$= e^{-\sqrt{n\lambda}\, t} \, M_{\overline{X}_n}\left(\sqrt{\frac{n}{\lambda}} \cdot t\right)$$

$$= e^{-\sqrt{n\lambda}\, t} \, e^{n\lambda\left(e^{\frac{t}{\sqrt{n\lambda}}} - 1\right)}$$

No send $n \to \infty$

take $\log_e$ first

$$\log M_{Z_n}(t) = -\sqrt{n\lambda}\cdot t + n\lambda\left(e^{\frac{t}{\sqrt{n\lambda}}} - 1\right)$$

expand the exponential part

recall $e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$

$\sqrt{n\lambda}\, t$

$$\log M_{Z_n}(t) = -\sqrt{n\lambda}\cdot t + n\lambda\left[\cancel{1} + \frac{t}{\sqrt{n\lambda}} + \frac{1}{2}\frac{t^2}{n\lambda} + \frac{1}{3!}\frac{t^3}{(n\lambda)^{3/2}} + \cdots \cancel{-1}\right]$$

$$= \frac{1}{2} t^2 + o\left(\frac{1}{\sqrt{n}}\right)$$

↑ going to zero

$\therefore$ with $n \to \infty$, we have

$$\log M_{Z_n}(t) \longrightarrow \frac{1}{2} t^2$$

so $M_{Z_n}(t) \longrightarrow e^{\frac{1}{2} t^2}$ This is the MGF $N(0,1)$

Thus $Z_n = \dfrac{\sqrt{n}(\bar{X}_n - \lambda)}{\sqrt{\lambda}} \xrightarrow{d} Z \sim N(0,1)$ $\square$ by Continuity Thm.

In the book Example A p. 181

$$X_n \sim \text{Poisson}(\lambda_n)$$

define $Z_n = \dfrac{X_n - E[X_n]}{\sqrt{V(X_n)}}$ ✓

Find the limiting distribution of Z_n as $n \to \infty$.

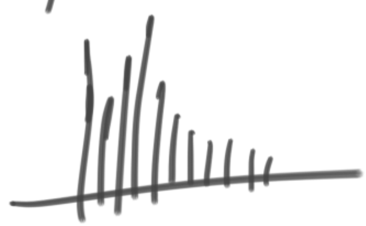
$E[X_n] = \lambda_n$

$\lambda_n \to \infty$

This is the limiting distribution of a Poisson r.v. X_n as its

$$E[X_n] = \lambda_n \to \infty$$

See p. 43 Figure 2.6



$\lambda = 1$

$\lambda = 5$

Normal Approximation to the Poisson.

In the book Example F p. 187

$X_n \sim Bin(n, p)$

$E[X_n] = np \to \infty$ $\quad (\lambda_n \to \infty)$

De Moivre – Laplace Limit Thm.

$$P\left(\frac{X_n - np}{\sqrt{np(1-p)}} \leq x \right) \to N(0,1)$$

as $n \to \infty$

Normal Approximation of the Binomial.

Central Limit Thm

Let $X_1, X_2, \ldots, X_n$ iid r.v.'s each have mean μ and variance σ^2.

Then the distribution of

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \longrightarrow Z \sim N(0,1)$$

where $S_n = \sum_{i=1}^{n} X_i$ tends to the standard normal.

$$\lim_{n \to \infty} P\left(\frac{S_n - n\mu}{\sigma \sqrt{n}} \le x \right) = \Phi(x)$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$

$-\infty < x < \infty$.

See Thm B p. 184

<u>pf</u>: Let $\mu = 0$.

Assume the $M_{X_n}(t)$ exists.

...

Thm: Let $F_1, F_2, \ldots$ be a sequence of CDF's and the corresponding PDF's be $f_1, f_2, \ldots$

If $\lim_{n \to \infty} f_n(x) = f(x) \ \forall x$, $f(x)$ is a PDF, then F_n converges in distribution to the CDF F corresponding to f.

See T-table.

<u>Example:</u> $X_n \sim t_n$.

PDF $$f_{X_n}(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}} \cdot \frac{1}{\sqrt{n}} \quad -\infty < x < \infty$$

Aside:
$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$= \frac{1}{\sqrt{2\pi}} \boxed{\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{\frac{n}{2}}}} \left(1+\frac{x^2}{n}\right)^{-n/2} \left(1+\frac{x^2}{n}\right)^{-\frac{1}{2}}$$

? Prob. 27 ?

as $n \to \infty$

$$f_{X_n}(t) \to \frac{1}{\sqrt{2\pi}} \cdot 1 \cdot e^{-\frac{x^2}{2}} \cdot 1 = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

PDF $N(0,1)$

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{\frac{n}{2}}} \longrightarrow 1$$

stirlings formula.

$$\Gamma(n+1) = n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n} =$$

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

try it with some numbers.

Prediction $E[Y|X]$. see Handout

see p. 152 Sec 4.4.2

The goal of Machine Learning is Prediction.

Next Derived Distributions Ch. 6.

Derived Distributions. ch. 6

χ^2, t, F . . . Beta.

$\sim N$

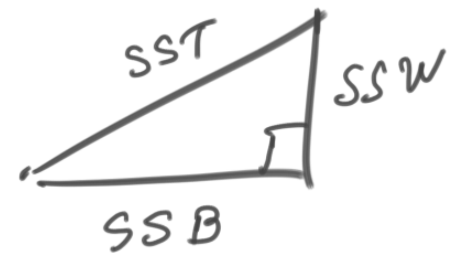
$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}}$$

$\sqrt{\chi^2}$ indep.

χ^2

$$F = \frac{SSB/df}{SSW/df}$$

χ^2 indep.



$\underline{\text{Def}}$: $Z \sim N(0,1)$, $U = Z^2 \sim \chi^2_1 = \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

p. 61

$\underline{\text{Def}}$: $U_1, \ldots, U_n$ iid χ^2

$$V = \sum_{i=1}^{n} U_i \sim \chi^2_n = \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$$

$$f_V(v) = \frac{1}{2^{n/2}\,\Gamma(\frac{n}{2})} v^{\frac{n}{2}-1} e^{-\frac{v}{2}} \qquad v > 0.$$

$$M_V(t) = (1-2t)^{-\frac{n}{2}} \qquad E[V] = n \qquad Var(V) = 2n$$

Aside: w/ data $\sigma^2 = \dfrac{\Sigma(x_i - \mu)^2}{n}$, $s^2 = \dfrac{\Sigma(x_i - \bar{x})^2}{n-1}$

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$(X_1 - \mu)^2 + (X_2 - \mu)^2 + \cdots + (X_n - \mu)^2$ sums of errors.

$\sigma^2 = \dfrac{\Sigma(X_i - \mu)^2}{n}$ $\quad S^2 = \dfrac{\Sigma(X_i - \bar{X})^2}{n-1}$

r.v. $\quad$ dist ??

Recall $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$