Chebyshev and Markov Inequalities

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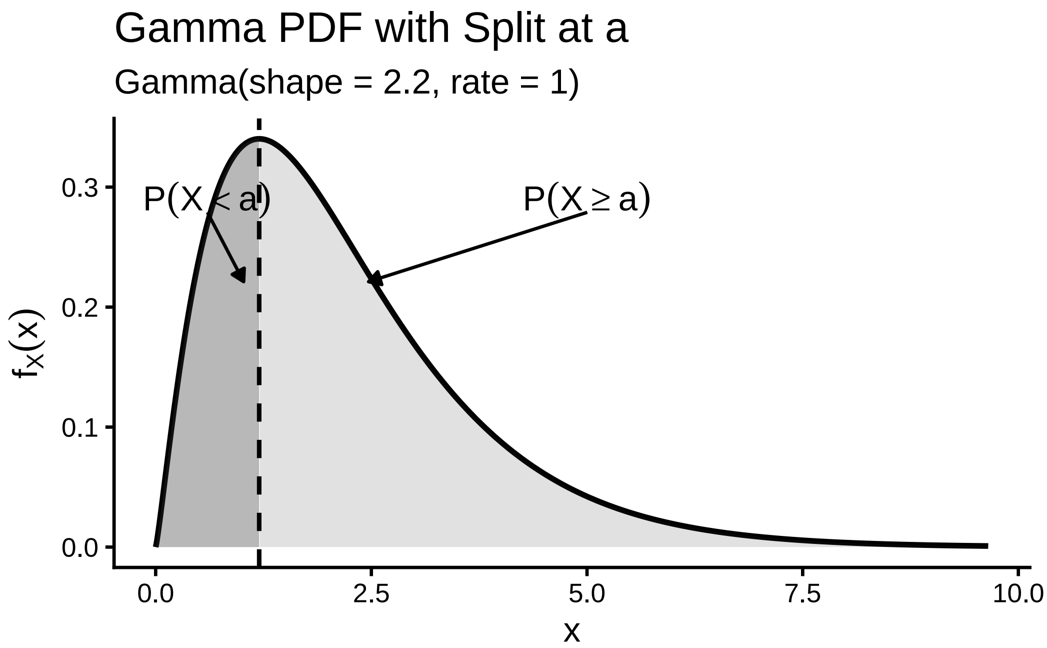
Probability Inequalities

Markov's Inequality

For a positive random variable X and any $a > 0$, we cannot put unlimited probability in the right tail. The precise statement is:

$$\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

Proof idea.



Gamma PDF with Split at a
Gamma(shape = 2.2, rate = 1)
P(X < a)
P(X ≥ a)
0.3
0.2
0.1
0.0
fX(x)
0.0
2.5
5.0
7.5
10.0
x

Define the indicator function as

$$I = 1 \quad X \geq a$$
$$I = 0 \quad X < a$$

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|I]] = \mathbb{E}[g(I)] = \mathbb{E}[X|I=0]Pr(X < a) + \mathbb{E}[X|I=1]Pr(X \geq a).$$

This is an application of the **Law of Total Expectation**.

Note: $P(X < a) \geq 0$ and $\mathbb{E}[X|I=0] \geq 0$ since $X > 0$.

So

$$\mathbb{E}[X] \geq \mathbb{E}[X|I=1]Pr(X \geq a)$$

When $I = 1$, $X \geq a$. So

$$\mathbb{E}[X] \geq \mathbb{E}[a|I=1]Pr(X \geq a) = aPr(X \geq a)$$

Therefore,

$$\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

An equivalent parameterization sets $a = b\mathbb{E}[X]$ (with $b > 0$), which yields

$$\Pr(X \geq b\,\mathbb{E}[X]) \leq \frac{1}{b}.$$

Some quick consequences (from the notes):

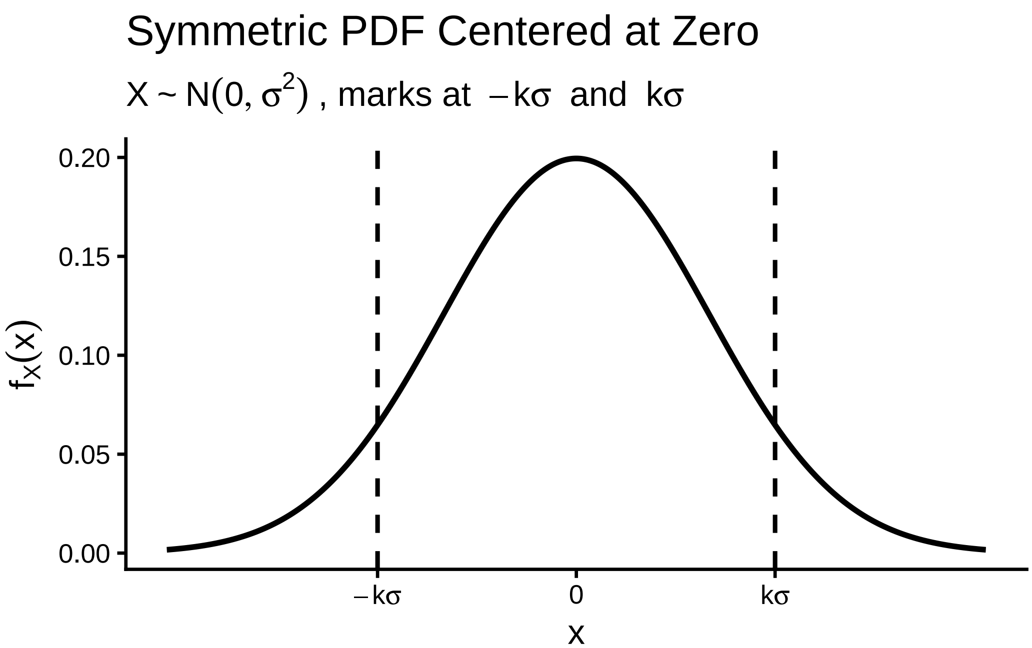
b	Bound on $P(X \geq b\mu)$
1	$P(X \geq \mu) \leq 1$
2	$P(X \geq 2\mu) \leq \frac{1}{2}$
3	$P(X \geq 3\mu) \leq \frac{1}{3}$

Chebyshev's Inequality

For any random variable X with finite mean μ and finite variance σ^2, measurement error is bounded in the sense that for $k > 0$:

$$\Pr\big(|X - \mu| \geq k\,\sigma\big) \leq \frac{1}{k^2}.$$

Proof via Markov.



Let $Y = (X - \mu)^2$, which is nonnegative. By Markov's inequality,

$$\Pr\big(Y \geq k^2\sigma^2\big) \leq \frac{\mathbb{E}[Y]}{k^2\sigma^2} = \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}.$$

Since $\{Y \geq k^2\sigma^2\} \equiv \{|X - \mu| \geq k\sigma\}$, the Chebyshev bound follows.

Equivalently, writing $k = b$ gives

$$\Pr\big(|X - \mu| \geq b\,\sigma\big) \leq \frac{1}{b^2},$$

and hence

$$\Pr\big(|X - \mu| < b\,\sigma\big) \geq 1 - \frac{1}{b^2}.$$

Typical values (as in the notes):

b	$\Pr(\lvert X-\mu\rvert \geq b\sigma)$	$\Pr(\lvert X-\mu\rvert < b\sigma)$
1	≤ 1	≥ 0
2	$\leq \frac{1}{4}$	$\geq \frac{3}{4}$
3	$\leq \frac{1}{9}$	$\geq \frac{8}{9}$

One–point probability interval (single draw)

From Chebyshev's inequality,

$$\Pr(\mu - b\sigma < X < \mu + b\sigma) \geq 1 - \frac{1}{b^2}.$$

This rearranges the absolute deviation statement to a two–sided interval around μ.

Confidence interval for the mean with a single observation $(n = 1)$

Using the same bound,

$$\Pr(|X - \mu| < b\sigma) \geq 1 - \frac{1}{b^2},$$

which is the same interval as above.

So we are at least $1 - \frac{1}{b^2}$ percent confident that the interval $(x - b\sigma, x + b\sigma)$ contains μ.

Confidence interval for the mean of n i.i.d. observations (known σ)

Let $\bar{X}$ be the sample mean of $X_1, \dots, X_n$ with common mean μ and variance σ^2. Since $\mathrm{Var}(\bar{X}) = \sigma^2/n$, Chebyshev gives, for any $b > 0$,

$$\Pr(|\bar{X} - \mu| \leq b\,\frac{\sigma}{\sqrt{n}}) \geq 1 - \frac{1}{b^2}.$$

Equivalently, with probability at least $1 - \frac{1}{b^2}$, the interval

$$\left(\bar{X} - b\,\frac{\sigma}{\sqrt{n}},\ \bar{X} + b\,\frac{\sigma}{\sqrt{n}}\right)$$

contains μ.

Example (from the notes)

Taking $b = 2$ (and σ known), a Chebyshev interval

$$\left(\bar{X} - 2\,\frac{\sigma}{\sqrt{n}},\ \bar{X} + 2\,\frac{\sigma}{\sqrt{n}}\right)$$

has **at least**

$$1 - \frac{1}{2^2} = \frac{3}{4} = 75\%$$

confidence.

> **Note:** Chebyshev bounds are distribution-free and can be very conservative.