

Order Statistics: When ordering a collection of independent continuous random variables with common cdf F (and pdf f) let

$$X_1, X_2, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} F \text{ (pdf } f)$$

and assume $f(x) \geq 0$ for $x \in (a, b)$ and 0 otherwise, then we define the order statistics as

$$X_{(1)} = \min \{ X_1, \dots, X_n \}.$$

\vdots

$$X_{(n)} = \max \{ X_1, \dots, X_n \}.$$

CDFs and pdfs.

$$F_{X_{(1)}}(x_i) = P(X_{(1)} \leq x_i) = 1 - P(X_{(1)} > x_i)$$

$$= 1 - P(\text{all } X_i > x_i)$$

$$= 1 - [P(X_1 > x_i) \cdot P(X_2 > x_i) \cdots P(X_n > x_i)]$$

$$= 1 - \left[\prod_{i=1}^n P(X_i > x_i) \right]$$

$$= 1 - \left[\prod_{i=1}^n (1 - F(x_i)) \right]$$

$$= 1 - [1 - F(x_i)]^n I_{(a,b)}(x).$$

$$f_{X_{(1)}}(x_1) = -n [1 - F(x_1)]^{n-1} (-f(x_1))$$

$$= n f(x_1) [1 - F(x_1)]^{n-1} I_{(a,b)}(x)$$

$$F_{X_{(n)}}(x_n) = P(X_{(n)} \leq x_n) = P(\text{all } X_i \leq x_n)$$

$$= P(X_1 \leq x_n) \cdot P(X_2 \leq x_n) \cdots P(X_n \leq x_n)$$

$$= \left[\prod_{i=1}^n P(X_i \leq x_n) \right] = \left[\prod_{i=1}^n F(x_n) \right]$$

$$= [F(x_n)]^n I_{(a,b)}(x_n)$$

$$f_{X_{(n)}}(x_n) = n [F(x_n)]^{n-1} f(x_n) I_{(a,b)}(x_n)$$

Example: $X_1, \dots, X_n \stackrel{iid}{\sim} U[0,1]$.

$$f(x) = I_{(0,1)}(x) \quad F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

so $f_{X_{(1)}}(x_1) = n [1 - x_1]^{n-1} I_{(0,1)}(x_1)$

and $f_{X_{(n)}}(x_n) = n x_n^{n-1} I_{(0,1)}(x_n)$.

Example: $X_1, \dots, X_n \sim \text{exp}(\lambda)$

$$f(x) = \lambda e^{-\lambda x} I_{(0, \infty)}(x)$$

$$F(x) = (1 - e^{-\lambda x}) I_{(0, \infty)}(x)$$

$$\begin{aligned} \text{so } f_{X_{(1)}}(x_1) &= n \lambda e^{-\lambda x_1} [e^{-\lambda x_1}]^{n-1} \\ &= n \lambda e^{-n \lambda x_1} I_{(0, \infty)}(x_1) \end{aligned}$$

$$\Rightarrow X_{(1)} \sim \text{Exp}(n\lambda)$$

$$\text{and } f_{X_{(n)}}(x_n) = n [1 - e^{-\lambda x_n}]^{n-1} e^{-\lambda x_n} I_{(0, \infty)}(x_n)$$

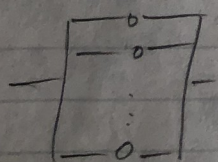
Example: $\underbrace{\quad \overset{0}{\circ} \quad \overset{0}{\circ} \quad \dots \quad \overset{0}{\circ} \quad}_{x \quad x \quad \quad \quad x} \quad X_i \sim \text{Exp}(\lambda)$

$$P(\text{lasts longer than } t) = P(X_{(1)} > t)$$

$$= 1 - P(X_{(1)} \leq t) = 1 - F_{X_{(1)}}(t)$$

$$= 1 - \left\{ 1 - [1 - (1 - e^{-\lambda t})]^n \right\}$$

$$= e^{-n \lambda t} I_{(0, \infty)}(t)$$



$$\begin{aligned} &P(\text{lasts longer than } t) \\ &= 1 - P(\text{lasts less than } t) \\ &= 1 - P(X_{(n)} \leq t) = 1 - F_{X_{(n)}}(t) \\ &= 1 - \left\{ [1 - e^{-\lambda t}]^n \right\} I_{(0, \infty)}(t) \end{aligned}$$

The density of the k^{th} order statistic is

$$f_{X_{(k)}}(x_k) = \frac{n!}{(k-1)!(n-k)!} [F(x_k)]^{k-1} f(x_k) \cdot [1-F(x_k)]^{n-k} I_{(a,b)}(x_k)$$

The joint density of the j^{th} and k^{th} order statistics is

$$f_{X_{(j)}, X_{(k)}}(x_j, x_k) = \frac{n!}{(j-1)!(k-j-1)!(n-k)!} \cdot [F(x_j)]^{j-1} f(x_j) \cdot [F(x_k) - F(x_j)]^{k-j-1} f(x_k) \cdot [1-F(x_k)]^{n-k} I_{(a,b)}(x_j) \cdot I_{(x_j, b)}(x_k)$$

Example: For $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} U(0,1)$ find the density of the k^{th} order statistic

$$f_{X_{(k)}}(x_k) = \frac{n!}{(k-1)!(n-k)!} x_k^{k-1} (1-x_k)^{n-k} I_{(0,1)}(x_k)$$

$$\Rightarrow X_{(k)} \sim \text{Beta}(k, n-k+1)$$

The joint density of the min and max

$$P_{x_1, x_n}^{x_1, x_n} = \frac{n!}{0!(n-1-1)!0!} [x_1]^{1-1} \cdot 1$$

$$[x_n - x_1]^{n-1-1} \cdot 1$$

$$[1 - x_n]^{n-n}$$

$$= n(n-1) (x_n - x_1)^{n-2} I_{(0,1)}(x_1) \cdot$$

$$I_{(x_1,1)}(x_n)$$