Order Statistics: When ordering a collection of independent continuous random variables with common cdf F (and pdf f) let

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} F \quad (\text{pdf } f)$$

and assume $f(x) \geq 0$ for $x \in (a, b)$ and 0 otherwise, then we define the order statistics as

$$X_{(1)} = \min\{X_1, \ldots, X_n\}.$$

$$\vdots$$

$$X_{(n)} = \max\{X_1, \ldots, X_n\}.$$

CDFs and pdfs.

$$F_{X_{(1)}}(x_1) = P(X_{(1)} \leq x_1) = 1 - P(X_{(1)} > x_1)$$

$$= 1 - P(\text{all } X_i > x_1)$$

$$= 1 - [P(X_1 > x_1) \cdot P(X_2 > x_1) \cdots P(X_n > x_1)]$$

$$= 1 - \left[\prod_{i=1}^{n} P(X_i > x_1)\right]$$

$$= 1 - \left[\prod_{i=1}^{n} (1 - F(x_1))\right]$$

$$= 1 - [1 - F(x_1)]^n \, I_{(a,b)}(x).$$

$$f_{X_{(1)}}(x_1) = -n\left[1 - F(x_1)\right]^{n-1}\left(-f(x_1)\right)$$

$$= n\, f(x_1)\left[1 - F(x_1)\right]^{n-1} I_{(a,b)}(x)$$

$$F_{X_{(n)}}(x_n) = P(X_{(n)} \le x_n) = P(\text{all } X_i \le x_n)$$

$$= P(X_1 \le x_n)\cdot P(X_2 \le x_n)\cdots P(X_n \le n)$$

$$= \left[\prod_{i=}^{n} P(X_i \le x_n)\right] = \left[\prod_{i=1}^{n} F(x_n)\right].$$

$$= \left[F(x_n)\right]^n I_{(a,b)}(x_n)$$

$$f_{X_{(n)}}(x_n) = n\left[F(x_n)\right]^{n-1} f(x_n)\, I_{(a,b)}(x_n)$$

<u>Example:</u> $X_1, \ldots, X_n \overset{iid}{\sim} U[0,1]$.

$$f(x) = I_{(0,1)}(x) \qquad F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le \\ 1 & x > 1 \end{cases}$$

so $f_{X_{(1)}}(x_1) = n\left[1 - x_1\right]^{n-1} I_{(0,1)}(x_1)$

and $f_{X_{(n)}}(x_n) = n\, x_n^{n}\, I_{(0,1)}(x_n)$.

Example: $X_1, \ldots, X_n \sim \exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$$

$$F(x) = (1 - e^{-\lambda x}) I_{(0,\infty)}(x)$$

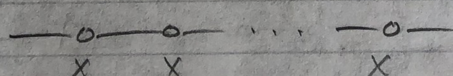
so $f_{X_{(1)}}(x_1) = n\lambda e^{-\lambda x_1}\left[e^{-\lambda x_1}\right]^{n-1}$

$$= n\lambda e^{-n\lambda x_1} I_{(0,\infty)}(x_1)$$

$$\Rightarrow X_{(1)} \sim Exp(n\lambda).$$

and $f_{X_{(n)}}(x_n) = n\left[1 - e^{-\lambda x_n}\right]^{n-1} e^{-\lambda x_n} I_{(0,\infty)}(x_n)$

Example:



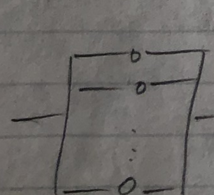
$X_i \sim Exp(\lambda)$

$P(\text{lasts longer than } t) = P(X_{(1)} > t)$

$$= 1 - P(X_{(1)} \le t) = 1 - F_{X_{(1)}}(t)$$

$$= 1 - \left\{1 - \left[1 - (1 - e^{-\lambda t})\right]^n\right\}.$$

$$= e^{-n\lambda t} I_{(0,\infty)}(t)$$



$P(\text{lasts longer than } t)$

$= 1 - P(\text{lasts less than } t)$

$= 1 - P(X_{(n)} \le t) = 1 - F_{X_{(n)}}(t)$

$$= 1 - \left[1 - e^{-\lambda t}\right]^n I_{(0,\infty)}(t)$$

The density of the k^{th} order statistic is

$$f_{X_{(k)}}(x_k) = \frac{n!}{(k-1)!\,(n-k)!}\left[F(x_k)\right]^{k-1} f(x_k) \cdot \left[1-F(x_k)\right]^{n-k} I_{(a,b)}(x_k)$$

The joint density of the j^{th} and k^{th} order statistics is

$$f_{X_{(j)},X_{(k)}}(x_j, x_k) = \frac{n!}{(j-1)!\,(k-j-1)!\,(n-k)!} \cdot \left[F(x_j)\right]^{j-1} f(x_j) \cdot \left[F(x_k) - F(x_j)\right]^{k-j-1} f(x_k) \cdot \left[1-F(x_k)\right]^{n-k} I_{(a,b)}(x_j) \cdot I_{(x_j,b)}(x_k)$$

Example: For $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathcal{U}(0,1)$ find the density of the k^{th} order statistic

$$f_{X_{(k)}}(x_k) = \frac{n!}{(k-1)!\,(n-k)!}\, x_k^{k-1} (1-x_k)^{n-k} I_{(0,1)}(x_k)$$

$$\Rightarrow X_{(k)} \sim Beta(k, n-k+1)$$

The joint density of the min and max

$$f_{X_{(1)},X_{(n)}}(x_1, x_n) = \frac{n!}{0!\,(n-1-1)!\,0!}\,[x_1]^{1-1}\cdot 1$$

$$[x_n - x_1]^{n-1-1}\cdot 1$$

$$[1 - x_n]^{n-n}$$

$$= n(n-1)\,(x_n - x_1)^{n-2}\; I_{(0,1)}(x_1)\cdot$$

$$I_{(x_1,1)}(x_n)$$