Functions of Several Variables

Suppose (X, Y), jointly distributed are mapped onto (U, V), jointly distributed.

$$u = g_1(x, y)$$
$$v = g_2(x, y)$$

and that the transformation can be inverted

$$x = h_1(u, v)$$
$$y = h_2(u, v).$$

Assume h_1 & h_2 have continuous partial derivatives and that the Jacobian

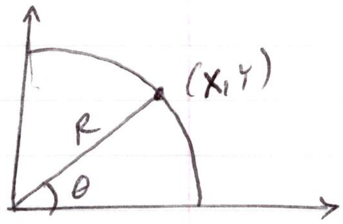
$$J = \begin{vmatrix} \frac{\partial}{\partial u} h_1 & \frac{\partial}{\partial v} h_1 \\ \frac{\partial}{\partial u} h_2 & \frac{\partial}{\partial v} h_2 \end{vmatrix} \neq 0$$

Then

$$f_{UV}(u, v) = f_{XY}(h_1(u, v), h_2(u, v)) \cdot |J|$$

<u>Example</u>: Let X and Y be independent $N(0,1)$ random variables, find the joint density of the polar coordinates R and Θ.

<u>recall</u>: polar coordinates



$$r = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}(y/x)$$

and

$$x = r\cos\theta$$
$$y = r\sin\theta$$

is the inverse transformation.

So $f_{R\Theta}(r,\theta) = f_{XY}(x,y)\,|J|$

find J

$$J = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & -r\cos\theta \end{bmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

$\therefore\ f_{R\Theta}(r,\theta) = r\, f_{XY}(r\cos\theta, r\sin\theta).$

$$= r \, f_X(r\cos\theta) \, f_Y(r\sin\theta)$$

$$= r \, \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2\cos^2\theta}{2}} \, \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2\sin^2\theta}{2}}$$

$$= \frac{r}{2\pi} e^{-\frac{r^2}{2}}$$

$$= f_R(r) \, f_\Theta(\theta).$$

where $f_R(r) = r e^{-\frac{r^2}{2}}$ $\qquad f_\Theta(\theta) = \frac{1}{2\pi}$

Rayleigh $\qquad$ Uniform$(0, 2\pi)$.

<u>Box - Muller Method</u>: Generate standard normal random values

1. Generate U_1, U_2 indep Unif$(0,1)$
2. $-2\log U_1 \sim \text{Exp}\left(\frac{1}{2}\right)$.

   $2\pi U_2 \sim \text{Unif}(0, 2\pi)$
3. $X = \sqrt{-2\log U_1} \, \cos(2\pi U_2) \sim N(0,1)$

   $Y = \sqrt{-2\log U_1} \, \sin(2\pi U_2) \sim N(0,1)$