Stat. 640: Final

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Instructions: This is a closed-book, closed internet, closed computer (no use of Python or R) and closed online communication exam. You may use 5 pages of notes, both sides. All work on the final is to be completed individually. Questions can be asked of the instructor of the course or by Zoom Chat. You may use a calculator, not an app on your phone.

Write your answers on your own paper in order. Show your work for full credit. Write your name on each page that is submitted.

You have the full class time, 2 hours.

1. **(20 points)** Let $X_1, X_2, ..., X_n$ be i.i.d. random variables from the Exponential distribution with mean θ. Determine the MLE, the Asymptotic Variance of the MLE, and specify the asymptotic 95% confidence interval for the unknown parameter of the Exponential distribution.

2. **(20 points)** Let $X_1, X_2, ..., X_n$ be i.i.d. random variables from the Pareto distribution having shape parameter α with density

$$f(x) = \alpha 2^{\alpha} x^{-(\alpha+1)} \quad \alpha > 0,\ x > 2$$

and 0 otherwise.

   a) Find the joint density of $X_1, X_2, ..., X_n$. This is the Likelihood function of α.
   b) Calculate the maximum likelihood estimator, $\hat{\alpha}$, of α.
   c) Find the asymptotic variance of the maximum likelihood estimator $\hat{\alpha}$.
   d) Give an approximate 95% confidence interval for α based on the maximum likelihood estimator $\hat{\alpha}$.

3. **(10 points)** Let $X_1, X_2, ..., X_n$ be i.i.d. random variables from the Pareto distribution having shape parameter α with density

$$f(x) = \alpha 2^{\alpha} x^{-(\alpha+1)} \quad \alpha > 0,\ x > 2$$

and 0 otherwise.

   a) Derive the likelihood ratio test of the simple hypotheses $H_0 : \alpha = \alpha_0$ versus $H_1 : \alpha = \alpha_1$ where $\alpha_0 < \alpha_1$ are specific numbers. Simplify the test statistic as much as possible. **Hint:** Use the Neyman-Person Lemma.
   b) Is the test uniformly most powerful against the alternative $H_1 : \alpha > \alpha_0$? Why or why not?

4. **(10 points)** Let $X_1, X_2, ..., X_n$ be i.i.d. random variables from the Pareto distribution having shape parameter α and scale parameter θ with density

$$f(x) = \alpha\theta^{\alpha}(x+\theta)^{-(\alpha+1)} \quad \alpha > 0,\ x > 0$$

and 0 otherwise. Using the output for a simulated dataset from this Pareto distribution, answer the following questions:

a) What are the maximum likelihood estimates of α and θ?
b) What are the approximate 95% confidence interval for α and θ?
c) **(Extra Credit)** Using the approximate confidence intervals, test the hypotheses that $H_0 : \alpha = 10$ and $H_0 : \theta = 2$.

```r
suppressMessages(library(actuar))

alpha <- 10
theta <- 2
n <- 3000

x.orig <- rpareto(n, shape = alpha, scale = theta)
```

```r
suppressMessages(library(fitdistrplus))

fit.pareto <- fitdist(x.orig, "pareto", method = "mle")
summary(fit.pareto)
```

```
Fitting of the distribution ' pareto ' by maximum likelihood
Parameters :
       estimate Std. Error
shape 10.878290  2.2763039
scale  2.253223  0.5128097
Loglikelihood:  1447.028   AIC:  -2890.056   BIC:  -2878.043
Correlation matrix:
          shape     scale
shape 1.0000000 0.9961864
scale 0.9961864 1.0000000
```