Stat. 640: Final

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Final

Instruction: This is an open-book and open-notes exam. This is a closed-internet and closed online communication exam. You may use a calculator. You may use a computer to access an ebook. Write your answers on your own paper. The exam paper will be available online through Canvas in the Files folder during your class time. The exam is an online Assignment in Canvas and needs to be submitted in Canvas at the end of the test time.

1. **(20 points)** Let $X_1, X_2, ..., X_n$ be i.i.d. random variables from the Pareto distribution with density

$$f(x) = \beta x^{-(\beta+1)} \quad \beta > 0,\ x > 1$$

   and 0 otherwise.

   a) Find the joint density of $X_1, X_2, ..., X_n$. This is the Likelihood function of β.
   b) Calculate the maximum likelihood estimator, $\hat{\beta}$, of β.
   c) Find the asymptotic variance of the maximum likelihood estimator $\hat{\beta}$.
   d) Give an approximate $100(1-\alpha)\%$ confidence interval for β based on the maximum likelihood estimator $\hat{\beta}$.

2. **(10 points)** Let $X_1, X_2, ..., X_n$ be i.i.d. random variables from the Pareto distribution with density

$$f(x) = \beta x^{-(\beta+1)} \quad \beta > 0,\ x > 1$$

   and 0 otherwise.

   a) Derive the likelihood ratio test of the hypothesis $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_1$ where β_0 and $\beta_0 < \beta_1$ are specific numbers. Simplify the test statistic as much as possible.
   b) Is the test uniformly most powerful against the alternative $H_1 : \beta > \beta_0$? Why or why not?

3. **(10 points)** Let $X_1, X_2, ..., X_n$ be i.i.d. random variables from the Pareto distribution with density

$$f(x) = \beta x^{-(\beta+1)} \quad \beta > 0, \ x > 1$$

and 0 otherwise.

a) Derive the generalized likelihood ratio test of the hypothesis $H_0 : \beta = \beta_0$ versus $H_1 : \beta \neq \beta_0$.
b) State the test using the large sample theory of the generalized likelihood ratio.

4. **(20 points)** Let X be a random variable whose probability mass function under H_0 and H_1 is given by

x	1	2	3	4	5	6	7
$f_0(x)$	.01	.01	.01	.01	.01	.01	.94
$f_1(x)$	.06	.05	.04	.03	.02	.01	.79

a) Accurately sketch a picture of the densities on the same axis.
b) Use the Neyman-Person Lemma to find the most powerful test for H_0 versus H_1 with size $\alpha = 0.05$.
c) Compute the power, π, of the test.
d) Compute the probability of Type II Error, β, for the test.

5. **(Extra Credit)** Let $X_1, X_2, ..., X_n$ be i.i.d. random variables from the Pareto distribution with density

$$f(x) = \beta x^{-(\beta+1)} \quad \beta > 0, \ x > 1$$

and 0 otherwise.

a) Calculate the expected value of $E[X]$, for $\beta > 1$.
b) Find the method of moments estimator, $\tilde{\beta}$, of β.
c) Explain how you would use the parametric bootstrap to compute an approximate $100(1-\alpha)\%$ confidence interval for β based on the method of moments estimator $\tilde{\beta}$.