Final Solution

Tuesday, April 7, 2026 9:40 AM

① $X_1, \ldots, X_n \sim Exp(\theta) \qquad E[X] = \theta$

a) $L(\theta) = f(x_1, \ldots, x_n | \theta) = \prod_{i=1}^{n} f(x_i | \theta)$

$$= \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} = \underline{\theta^{-n} e^{-\Sigma x_i/\theta}}$$

b) $\ell(\theta) = \log L(\theta) = -n \log \theta - \theta^{-1} \Sigma x_i$

$$\ell'(\theta) = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2}$$

$$\ell'(\theta) = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2} = 0$$

$$\frac{\Sigma x_i}{\theta^2} = \frac{n}{\theta}$$

$$\hat{\theta} = \frac{\Sigma x_i}{n} = \bar{X}$$

c) Determine AV of $\hat{\theta}$

$$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta}$$

$$\log f(x|\theta) = -\log \theta - \frac{x}{\theta}$$

$$\log f(x|\theta) = -\log(\theta) - \frac{x}{\theta}$$

$$\frac{d}{d\theta} \log f(x|\theta) = -\frac{1}{\theta} + \frac{x}{\theta^2}$$

$$\frac{d^2}{d\theta^2} \log f(x|\theta) = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$$

$$I(\theta) = -E\left[\frac{d^2}{d\theta^2} \log f(X|\theta)\right]$$

$$= -E\left[\frac{1}{\theta^2} - \frac{2X}{\theta^3}\right]$$

$$= -\frac{1}{\theta^2} + \frac{2}{\theta^3} E[X]$$

$$= -\frac{1}{\theta^2} + \frac{2}{\theta^3} \cdot \theta$$

$$= -\frac{1}{\theta^2} + \frac{2}{\theta^2} = \frac{1}{\theta^2}$$

$$AV(\hat{\theta}) = \frac{1}{n \cdot \frac{1}{\theta^2}} = \frac{\theta^2}{n}$$

d) 95% CI for θ

$$\hat{\theta} \pm z_{\alpha/2} \sqrt{\widehat{AV}(\hat{\theta})}$$

$$\hat{\theta} \pm z_{\alpha/2} \sqrt{\frac{\hat{\theta}^2}{n}}$$

$$\hat{\theta} \pm z_{\alpha/2} \frac{\hat{\theta}}{\sqrt{n}}$$

(2) $X_1, \dots, X_n \sim \text{Pareto}(\beta, 3)$

$$f(x|\beta) = \beta \, 3^{\beta} \, x^{-(\beta+1)} \qquad \beta > 0, \quad x > 3$$

a) $$L(\beta) = f(x_1, \dots, x_n|\beta) = \prod_{i=1}^{n} f(x_i|\beta)$$

$$= \prod_{i=1}^{n} \beta \, 3^{\beta} \, x_i^{-(\beta+1)}$$

$$= \beta^n \, 3^{n\beta} \left(\prod_{i=1}^{n} x_i\right)^{-(\beta+1)}$$

b) $H_0: \beta = \beta_0$

$H_1: \beta = \beta_1 \qquad \beta_0 < \beta_1$

$$\Lambda = \frac{L(\beta_0)}{L(\beta_1)} = \frac{\beta_0^n \, 3^{n\beta_0} \left(\prod x_i\right)^{-(\beta_0+1)}}{\beta_1^n \, 3^{n\beta_1} \left(\prod x_i\right)^{-(\beta_1+1)}}$$

$$= \left(\frac{\beta_0}{\beta_1}\right)^n 3^{n(\beta_0-\beta_1)} \left(\prod x_i\right)^{-\beta_0-1+\beta_1+1}$$

$$= \left(\frac{\beta_0}{\beta_1}\right)^n 3^{n(\beta_0-\beta_1)} \left(\prod x_i\right)^{\beta_1-\beta_0}$$

NP Lemma Reject H_0 if $\Lambda < k$

$$\left(\frac{\beta_0}{\beta_1}\right)^n 3^{n(\beta_0-\beta_1)} \left(\prod x_i\right)^{\beta_1-\beta_0} < k$$

$$\left(\prod x_i\right)^{\beta_1-\beta_0} < k'$$

since $\beta_0, \beta_1 > 0$ $\quad \frac{\beta_0}{\beta_1} > 0$, $\quad 3^{n(\beta_0-\beta_1)} > 0$

$$\prod x_i < k''$$

since $\beta_1 - \beta_0 > 0$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \prod x_i < c \\ 0 & \text{if } \prod x_i \geq c \end{cases}$$

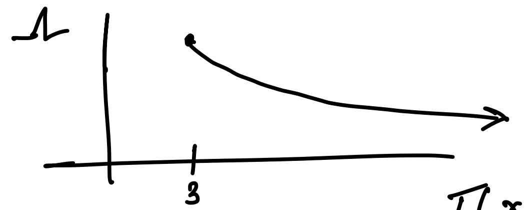
c) For the test to be uniformly most powerful it must have MLR in the statistic

$\beta_1 < \beta_2$

switch the order.

$$\frac{L(\beta_2)}{L(\beta_1)} = \left(\frac{\beta_2}{\beta_1}\right)^n 3^{n(\beta_1-\beta_2)} \left(\prod x_i\right)^{-(\beta_1-\beta_2)}$$

$T(\underset{\sim}{x}) = \prod x_i$



$T(x) = \prod x_i$

Yes. $\quad \phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \prod x_i < c \\ 0 & \text{if } \prod x_i \geq c \end{cases}$ UMP

for $\quad H_0: \beta = \beta_0$

$\qquad H_1: \beta > \beta_0$

(3) $\quad X_1, \ldots, X_n \sim \text{Pareto}(\beta, 3)$

$$f(x \mid \beta) = \beta\, 3^{\beta} x^{-(\beta+1)} \qquad \beta > 0,\ x > 3$$

a) $\Lambda^* = \dfrac{\max_{\Omega_0} L(\beta)}{\max_{\Omega} L(\beta)} = \dfrac{L(\beta_0)}{L(\hat{\beta})}$

$$= \frac{\beta_0\, 3^{n\beta_0} \left(\prod x_i\right)^{-(\beta_0+1)}}{\hat{\beta}\, 3^{n\hat{\beta}} \left(\prod x_i\right)^{-(\hat{\beta}+1)}}$$

$$\phi^*(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \Lambda^* < k \\ 0 & \text{otherwise} \end{cases}$$

b) $$\phi^*(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } -2\log\Lambda^* > c \\ 0 & \text{otherwise} \end{cases}$$

4) $X_1, \ldots, X_n \sim \text{Pareto}(\alpha, \theta)$

$$f(x \mid \alpha, \theta) = \alpha\,\theta^{\alpha}(x+\theta)^{-(\alpha+1)} \quad \alpha > 0,\ x > \theta$$

a) $\hat{\alpha} = 2.96 \qquad \hat{\theta} = 14.58$

b) 95% CI for α

$$\hat{\alpha} \pm 1.96 \cdot SE(\hat{\alpha})$$

$$2.96 \pm 1.96\,(.21)$$

$$(2.55,\ 3.37)$$

95% CI for θ

$$\hat{\theta} \pm 1.96 \cdot SE(\hat{\theta})$$

$$\hat{\theta} \pm 1.96 \cdot SE(\hat{\theta})$$

$$14.58 \pm 1.96\,(1.34)$$

$$(11.95,\ 17.21)$$

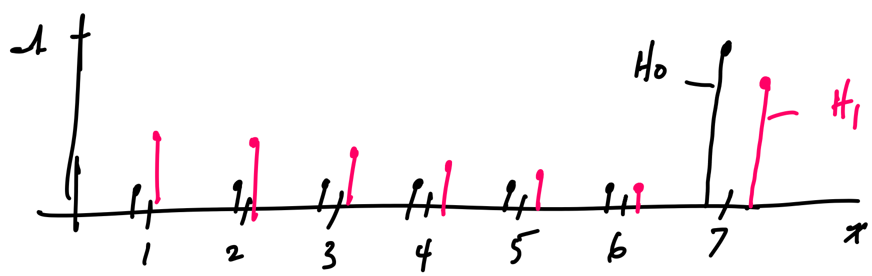
c) $H_0: \alpha = 10$ Reject H_0

$H_0: \theta = 3$ Reject H_0

⑤ Extra Credit

x	1	2	3	4	5	6	7
$f_0(x)$	.01	.01	.01	.01	.01	.01	.94
$f_1(x)$	.06	.05	.04	.03	.02	.01	.79
λ	$1/6$	$1/5$	$1/4$	$1/3$	$1/2$	1	$\frac{94}{79}$

a)



b) Reject H_0 if $\lambda < c$ order $x = 1,2,3,4,5$

$$\phi(x) = \begin{cases} 1 & \text{if } x = 1,2,3,4,5 \\ 0 & \text{if } x = 6,7 \end{cases}$$

c) $\pi = P_1(x \in R) = P_1(\phi(x) = 1)$

$= P_1(x = 1,2,3,4,5) = .20$

d) $\beta = 1 - \pi = 1 - .20 = .80$