Stat. 640: Final

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Instruction: Show your work where appropriate. Each part is worth **5 points**. This is an closed-book exam. You may use 20 pages of notes that you have prepared. This is a closed-internet and closed online communication exam. You may use a calculator. You may not use a phone or computer. You may use a calculator (not a phone). Questions should be directed to the instructor. You are expected to be academically honest while taking the test. Show your work for full credit.

Submit your test paper, with your name on it, and your handwritten work in the order of the test. Staple you test and papers together. Submit the test paper with your papers in order folded in half longways.

1. (**20 points**) Let $X_1, ..., X_n$ be i.i.d. random variables from the Exponential distribution with mean θ.

   a) Find the joint density of $X_1, X_2, ..., X_n$. This is the Likelihood function of θ.
   b) Determine the MLE, $\hat{\theta}$, of θ.
   c) Determine the Asymptotic Variance of the MLE, $\hat{\theta}$.
   d) Specify the asymptotic 95% confidence interval for the unknown parameter of the Exponential distribution.

2. (**15 points**) Let $X_1, X_2, ..., X_n$ be i.i.d. random variables from the Pareto distribution with density

$$f(x) = \beta\ 3^{\beta}\ x^{-(\beta+1)} \quad \beta > 0,\ x > 3$$

   and 0 otherwise.

   a) Find the joint density of $X_1, X_2, ..., X_n$. This is the Likelihood function of β.
   b) Derive the likelihood ratio test of the hypothesis $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_1$ where $\beta_0 < \beta_1$ are specific numbers. Simplify the test statistic as much as possible. **Hint:** Use the Neyman-Person Lemma.
   c) Is the test uniformly most powerful against the alternative $H_1 : \beta > \beta_0$? Why or why not? Justify your answer.

3. (**10 points**) Let $X_1, X_2, ..., X_n$ be i.i.d. random variables from the Pareto distribution with density

$$f(x) = \beta\ 3^{\beta}\ x^{-(\beta+1)} \quad \beta > 0,\ x > 3$$

   and 0 otherwise.

   a) Derive the generalized likelihood ratio test of the hypothesis $H_0 : \beta = \beta_0$ versus $H_1 : \beta \neq \beta_0$.
   b) State the test using the large sample theory of the generalized likelihood ratio.

4. **(15 points)** Let $X_1, ..., X_n$, be i.i.d. random variables from the Pareto distribution having shape parameter α and scale parameter θ with density

$$f(x) = \alpha\ \theta^{\alpha}\ (x+\theta)^{-(\alpha+1)} \quad \alpha > 0,\ x > \theta$$

and 0 otherwise. Using the output for a simulated dataset from this Pareto distribution, answer the following questions:

a) What are the maximum likelihood estimates of α and θ?
b) What are the approximate 95% confidence interval for α and θ?
c) Using the asymptotic confidence intervals, test the hypotheses that $H_0 : \alpha = 10$ and $H_0 : \theta = 3$.

```r
alpha <- 3
theta <- 15
n <- 3000

x.orig <- rpareto(n, shape = alpha, scale = theta)

fit.pareto <- fitdist(x.orig, "pareto", method = "mle")
summary(fit.pareto)
```

```
Fitting of the distribution ' pareto ' by maximum likelihood
Parameters :
       estimate Std. Error
shape  2.960716  0.2097739
scale 14.583224  1.3356852
Loglikelihood:  -8796.135   AIC:  17596.27   BIC:  17608.28
Correlation matrix:
          shape     scale
shape 1.0000000 0.9662297
scale 0.9662297 1.0000000
```

5. **(Extra Credit)** Let X be a random variable whose probability mass function under H_0 and H_1 is given by

x	1	2	3	4	5	6	7
$f_0(x)$	.01	.01	.01	.01	.01	.01	.94
$f_1(x)$	.06	.05	.04	.03	.02	.01	.79

a) Accurately sketch a picture of the densities on the same axis.
b) Use the Neyman-Person Lemma to find the most powerful test for H_0 versus H_1 with size $\alpha = 0.05$.
c) Compute the power, π, of the test.
d) Compute the probability of Type II Error, β, for the test.