Final Solution

Saturday, March 1, 2025   8:20 AM

1. $X_1, X_2, \ldots, X_n$ r.s. $Exp(\theta)$

$$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta} \qquad \theta > 0, \; x > 0$$

MLE

$$L(\theta) = f(x_1, \ldots, x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta}$$

$$= \theta^{-n} e^{-\Sigma x_i/\theta}$$

$$\ell(\theta) = \log L(\theta)$$

$$= -n \log \theta - \Sigma x_i/\theta = -n \log \theta - \Sigma x_i \cdot \theta^{-1}$$

$$\frac{d}{d\theta} \ell(\theta) = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2}$$

$\frac{d}{d\theta} \ell(\theta) = 0$ find maximum

$$-\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2} = 0$$

$$\frac{\Sigma x_i}{} = \frac{n}{}$$

$\theta^2 - \theta$

$\bar{X} = \theta \quad \therefore \hat{\theta} = \bar{X}$

Fisher's Information:

$$I(\theta) = -n\, E\left[\frac{d^2}{d\theta^2} \log f(x|\theta)\right]$$

start with the density

$$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta} = \theta^{-1} e^{-x/\theta}$$

$$\log f(x|\theta) = -\log \theta - \frac{x}{\theta} = -\log \theta - x\theta^{-1}$$

$$\frac{d}{d\theta} \log f(x|\theta) = -\frac{1}{\theta} + \frac{x}{\theta^2} = -\theta^{-1} + x\theta^{-2}$$

$$\frac{d^2}{d\theta^2} \log f(x|\theta) = \theta^{-2} - 2x\theta^{-3}$$

$$E\left[\frac{d^2}{d\theta^2} \log f(X|\theta)\right] = E\left[\theta^{-2} - 2X\theta^{-3}\right]$$

$$= \theta^{-2} - 2E[X]\theta^{-3}$$

$$= \theta^{-2} - 2\theta\theta^{-3}$$

$$= \theta^{-2} - 2\theta^{-2}$$

$$= \frac{1}{\theta^2} - \frac{2}{\theta^2}$$

$$= -\frac{1}{\theta^2}$$

$$I_X(\theta) = -E\left[-\frac{1}{\theta^2}\right] = \frac{1}{\theta^2}$$

single X

AV of the MLE

$$AV(\hat{\theta}) = \frac{1}{n I_X(\theta)} = \frac{\theta^2}{n}.$$

95% CI for θ

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{\widehat{AV}(\hat{\theta})}$$

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{\frac{\hat{\theta}^2}{n}}$$

$$\boxed{\hat{\theta} \pm z_{\alpha/2}\frac{\hat{\theta}}{\sqrt{n}}}$$

(2) $X_1, X_2, \ldots, X_n$ iid Pareto distributi

with shape parameter α

$$f(x|\alpha) = \alpha 2^{\alpha} x^{-(\alpha+1)} \qquad \begin{array}{l} \alpha > 0 \\ x > 2 \end{array}$$

a) joint density of $X_1, \dots, X_n$

$$L(\alpha) = f(x_1, \dots, x_n | \alpha) = \prod_{i=1}^{n} f(x_i | \alpha)$$

$$= \prod_{i=1}^{n} \alpha 2^{\alpha} x_i^{-(\alpha+1)}$$

$$= \alpha^n 2^{n\alpha} \left(\prod x_i\right)^{-(\alpha+1)}$$

b) MLE

$$l(\alpha) = \log L(\alpha)$$

$$= n \log(\alpha) + n\alpha \log(2) - (\alpha+1) \log\left(\prod x_i\right)$$

$$= n \log(\alpha) + n\alpha \log(2)$$

$$-(\alpha+1)\sum_{i=1}\log(x_i)$$

$$= n\log(\alpha) + n\alpha\log(2)$$

$$-\alpha\sum\log(x_i) - \sum\log(x_i)$$

$$\frac{d}{d\alpha}l(\alpha) = \frac{n}{\alpha} + n\log(2)$$

$$-\sum\log(x_i)$$

$\frac{d}{d\alpha}l(\alpha) = 0$ find maximum

$$\frac{n}{\alpha} + n\log(2) - \sum\log(x_i) = 0$$

$$\frac{1}{n}\cdot\frac{n}{\alpha} = \left(\sum\log(x_i) - n\log(2)\right)\frac{1}{n}$$

$$\frac{1}{\alpha} = \frac{1}{n}\left(\sum\log(x_i) - n\log(2)\right)$$

$$\hat{\alpha} = n\left(\sum\log(x_i) - n\log(2)\right)^{-1}$$

c) Fisher's Information:

$f(x|\alpha)$ density

$$f(x|\alpha) = \alpha\, 2^{\alpha}\, x^{-(\alpha+1)}$$

$$\log f(x|\alpha) = \log(\alpha) + \alpha\log(2) - (\alpha+1)\log(x)$$

$$= \log(\alpha) + \alpha\log(2) - \alpha\log(x) - \log(x)$$

$$\frac{d}{d\alpha}\log f(x|\alpha) = \frac{1}{\alpha} + \log(2) - \log(x)$$

$$\frac{d^2}{d\alpha^2}\log f(x|\alpha) = -\frac{1}{\alpha^2}$$

$$I(\alpha) = -E\left[\frac{d^2}{d\alpha^2}\log f(x|\alpha)\right]$$

$$= -E\left[-\frac{1}{\alpha^2}\right] = \frac{1}{\alpha^2}$$

AV

$$AV(\hat{\alpha}) = \frac{1}{nI(\alpha)} = \frac{\alpha^2}{n}$$

d) approximate 95% CI for α

$$\hat{\alpha} \pm z_{\alpha/2} \sqrt{\widehat{AV}(\hat{\alpha})}$$

$$\hat{\alpha} \pm z_{\alpha/2} \sqrt{\frac{\hat{\alpha}^2}{n}}$$

$$\boxed{\hat{\alpha} \pm z_{\alpha/2} \frac{\hat{\alpha}}{\sqrt{n}}}$$

③ $X_1, \dots, X_n$ r.s. Pareto distribution

$$f(x|\alpha) = \alpha\, 2^{\alpha} x_i^{-(\alpha+1)} \qquad \begin{array}{l} \alpha > 0 \\ x > 2 \end{array}$$

a) $H_0: \alpha = \alpha_0$

$H_1: \alpha = \alpha_1 \qquad \alpha_0 < \alpha_1$

N.P. Lemma

$$\Lambda = \frac{L(\alpha_0)}{L(\alpha_1)} = \frac{\alpha_0^n\, 2^{n\alpha_0} \left(\prod x_i\right)^{-(\alpha_0+1)}}{\alpha_1^n\, 2^{n\alpha_1} \left(\prod x_i\right)^{-(\alpha_1+1)}}$$

$$= \left(\frac{\alpha_0}{\alpha_1}\right)^n 2^{n(\alpha_0-\alpha_1)} \left(\prod x_i\right)^{-(\alpha_0+1)+(\alpha_1+1)}$$

$$= \left(\frac{\alpha_0}{\alpha_1}\right)^n 2^{n(\alpha_0-\alpha_1)} \left(\prod x_i\right)^{\alpha_1-\alpha_0}$$

Reject H_0 $\Lambda < k$

$$\left(\frac{\alpha_0}{\alpha_1}\right)^n 2^{n(\alpha_0-\alpha_1)} \left(\prod x_i\right)^{\alpha_1-\alpha_0} < k$$

since $\frac{\alpha_0}{\alpha_1} < 1$

but $\left(\frac{\alpha_0}{\alpha_1}\right)^n > 0$,

since $\alpha_1 - \alpha_0 > 0$

and

$2^{n(\alpha_0-\alpha_1)} < 1$

but $2^{n(\alpha_0-\alpha_1)} > 0$

$$\Longleftrightarrow \left(\prod x_i\right)^{\alpha_1-\alpha_0} < k'$$

$$\Longleftrightarrow \prod x_i < k''$$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \prod x_i < k \\ 0 & \text{otherwise} \end{cases}$$

b) Yes

For the test to be uniformly most powerful it must have MLR in the statistic.

$\alpha_1 < \alpha_2$

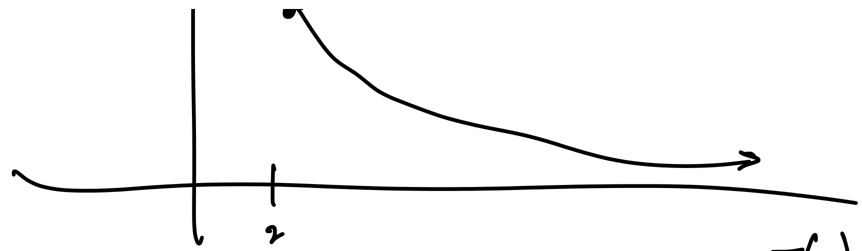
$\alpha_1 - \alpha_2$

$$\frac{L(\alpha_2)}{L(\alpha_1)} = \left(\frac{\alpha_2}{\alpha_1}\right)^n 2^{n(\alpha_2 - \alpha_1)} \left(\prod x_i\right)^{-(\alpha_2-\alpha_1)}$$

$$T(\underset{\sim}{X}) = \prod x_i$$

Since $\alpha_2 - \alpha_1 > 0$

Λ | Then $-(\alpha_2 - \alpha_1) < 0$



$T(\underset{\sim}{x}) = \prod x_i$

Monotone decreasing

So answer Yes

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \prod x_i < c \\ 0 & \text{otherwise} \end{cases} \qquad \text{UMP}$$

for $H_0: \alpha = \alpha_0$

$H_1: \alpha > \alpha_0$

(4) $X_1, X_2, \dots, X_n$ Pareto distribution
shape α, scale θ

$$f(x \mid \alpha, \theta) = \alpha \theta^n (x+\theta)^{-(\alpha+1)} \qquad \begin{array}{l} \alpha > 0 \\ \theta > 0 \end{array}$$

$x > 0$

a) $\hat{\alpha} = 10.8783$

$\hat{\theta} = 2.2532$

b) approximate 95% CI for α

$\hat{\alpha} \pm 1.96 \ se_{\hat{\alpha}}$

4.45

$10.8783 \pm 1.96(2.27)$

$(6.43, 15.33)$

approximate 95% CI for θ

$\hat{\theta} \pm 1.96 \ se_{\hat{\theta}}$

$2.2532 \pm 1.96(.513)$

$(1.25, 3.25)$

c. Extra Credit.

$H_0: \alpha = 10$ Fail to Reject H_0

$H_0: \theta = 2$ Fail to Reject H_0

both CIs contain the null value, so Fail to Reject each null hypothesis.