Stat 640 : Final

(1) $X_1, \dots, X_n$ iid Pareto

$$f(x) = \begin{cases} \beta x^{-(\beta+1)} & \beta > 0 \quad x > 1 \\ 0 & x \le 1 \end{cases}$$

a) $L(\beta) = f(x_1, \dots, x_n \mid \beta) = \prod_{i=1}^{n} f(x_i \mid \beta)$

$$= \prod^{n} \beta x_i^{-(\beta+1)} = \beta^n \left(\prod x_i\right)^{-(\beta+1)}$$

b) $l(\beta) = \log L(\beta) = n \log \beta - (\beta+1) \log\left(\prod x_i\right)$

$$= n \log \beta - (\beta+1) \sum^{n} \log(x_i)$$

$$= n \log \beta - \beta \sum \log(x_i) - \sum \log(x_i)$$

$\frac{d}{d\beta} l(\beta) = \frac{n}{\beta} - \sum_{i=1}^{n} \log(x_i) = 0$

$$\frac{n}{\beta} = \sum \log(x_i)$$

$$\hat{\beta} = \frac{n}{\sum \log(x_i)}$$

c) $AV(\hat{\beta}) = \frac{1}{nI(\beta)}$

$I(\beta)$ $\quad f(x \mid \beta) = \beta x^{-(\beta+1)}$

$$\log f(x \mid \beta) = \log(\beta) - (\beta+1) \log(x)$$

$$= \log(\beta) - \beta \log(x) - \log(x)$$

$$\frac{d}{d\beta} \log f(x|\beta) = \frac{1}{\beta} - \log(x)$$

$$\frac{d^2}{d\beta^2} \log f(x|\beta) = -\frac{1}{\beta^2}$$

$$I(\beta) = -E\left[\frac{d^2}{d\beta^2} \log f(X|\beta)\right]$$

$$= -E\left[-\frac{1}{\beta^2}\right]$$

$$= \frac{1}{\beta^2}$$

$$AV(\hat{\beta}) = \frac{1}{n\frac{1}{\beta^2}} = \frac{\beta^2}{n}.$$

d) $\hat{\beta} \pm z_{\alpha/2} \sqrt{\widehat{AV}(\beta)}$

$$\hat{\beta} \pm z_{\alpha/2} \cdot \frac{\hat{\beta}}{\sqrt{n}}$$

(2) $X_1, \ldots, X_n$ iid Pareto

$$f(x) = \begin{cases} \beta x^{-(\beta+1)} & \beta > 0,\ x > 1 \\ 0 & x \le 1 \end{cases}$$

a) $H_0: \beta = \beta_0$

$H_1: \beta = \beta_1 \qquad \beta_0 < \beta_1$

$$\Lambda = \frac{L(\beta_0)}{L(\beta_1)} = \frac{\beta_0^n \left(\prod x_i\right)^{-(\beta_0+1)}}{\beta_1^n \left(\prod x_i\right)^{-(\beta_1+1)}}$$

$$= \left(\frac{\beta_0}{\beta_1}\right)^n \left(\prod x_i\right)^{-(\beta_0+1)+(\beta_1+1)}$$

$$= \left(\frac{\beta_0}{\beta_1}\right)^n \left(\prod x_i\right)^{-\beta_0-1+\beta_1+1}$$

$$= \left(\frac{\beta_0}{\beta_1}\right)^n \left(\prod x_i\right)^{\beta_1-\beta_0}$$

$\Lambda < k$

$$\left(\frac{\beta_0}{\beta_1}\right)^n \left(\prod x_i\right)^{\beta_1-\beta_0} < k$$

$$\left(\prod x_i\right)^{\beta_1-\beta_1} < k' \qquad \text{since } \frac{\beta_0}{\beta_1} < 1$$

$$\prod x_i < k'' \qquad \text{but } \left(\frac{\beta_0}{\beta_1}\right)^n > 0$$

since $\beta_1 - \beta_0 > 0$

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \prod x_i < c \\ 0 & \text{if } \prod x_i \geq c \end{cases}$$

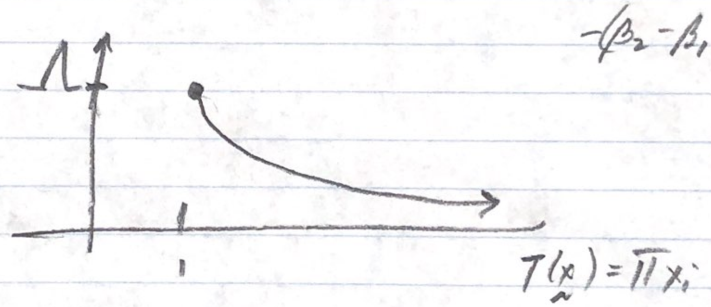
b) For the test to be uniformly most powerful it must have MLR in the statistic

$\beta_1 < \beta_2$

$$\frac{L(\beta_2)}{L(\beta_1)} = \left(\frac{\beta_2}{\beta_1}\right)^n \left(\prod^n x_i\right)^{-(\beta_2 - \beta_1)}$$

$T(\underset{\sim}{x}) = \prod x_i$

$-(\beta_2 - \beta_1) < 0$



monotone decreasing.

Yes $\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \prod x_i < c \\ 0 & \text{if } \quad \geq c \end{cases}$ UMP

for $H_0: \beta = \beta_0$

$H_1: \beta > \beta_0$

(3) $X_1, X_2, \ldots, X_n$ iid Pareto

$$f(x) = \begin{cases} \beta x^{-(\beta+1)} & \beta > 0,\ x > 1 \\ 0 & \text{otherwise} \end{cases}$$

a)

$$\Lambda^* = \frac{\max_{\Omega_0} L(\beta)}{\max_{\Omega} L(\beta)} = \frac{L(\beta_0)}{L(\hat{\beta})}$$

$$= \frac{\beta_0^n \left(\prod x_i\right)^{-(\beta_0+1)}}{\hat{\beta}^n \left(\prod x_i\right)^{-(\hat{\beta}+1)}}$$

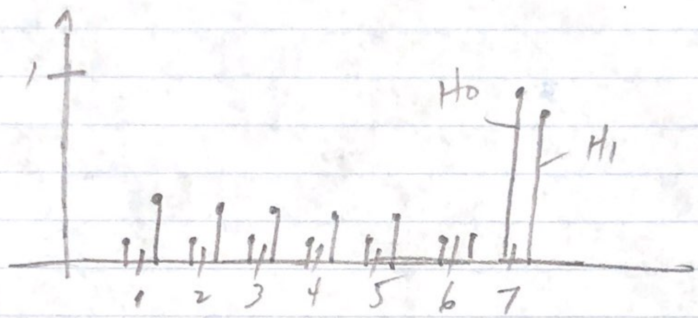
$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \Lambda^* < c \\ 0 & \text{otherwise} \end{cases}$$

b) $$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } -2\log \Lambda^* > \chi_1^2 \\ 0 & \text{otherwise} \end{cases}$$

(4)

x	1	2	3	4	5	6	7
$f_0(x)$	.01	.01	.01	.01	.01	.01	.94
$f_1(x)$	.06	.05	.04	.03	.02	.01	.79
Λ	1/6	1/5	1/4	1/3	1/2	1	94/79

a)



b) Reject H_0 $\Lambda < c$ order 1, 2, 3, 4, 5

$$\phi(x) = \begin{cases} 1 & \text{if } x = 1,2,3,4,5 \\ 0 & \text{if } x = 6,7 \end{cases}$$

c) $\pi = P_1(X \in R) = P_1(X = 1,2,3,4,5) = .20$

d) $\beta = 1 - \pi = 1 - .20 = .80$

⑤ $X_1, \dots X_n$ iid Pareto

$$f(x) = \begin{cases} \beta x^{-(\beta+1)} & \beta > 0, \quad x > 1 \\ 0 \end{cases}$$

a) $E[X] = \int_1^\infty x \, \beta x^{-(\beta+1)} \, dx$

$= \beta \int_1^\infty x^{-\beta} \, dx$

$= \beta \left. \dfrac{x^{-\beta+1}}{-\beta+1} \right|_1^\infty$

$= -\dfrac{\beta}{\beta-1} \left. x^{-\beta+1} \right|_1^\infty \qquad \beta > 1$

$= -\dfrac{\beta}{\beta-1} \left[0 - 1 \right] \qquad$ if $\beta < 1$

$E[X] = \infty$

$= \dfrac{\beta}{\beta-1}$

$$\frac{\beta}{\beta-1} = \bar{X}$$

$$\beta = \beta\bar{X} - \bar{X}$$

$$\bar{X} = \beta\bar{X} - \beta$$

$$\bar{X} = \beta(\bar{X} - 1)$$

$$\tilde{\beta} = \frac{\bar{X}}{\bar{X} - 1}$$

b) Parametric Bootstrap

Take $B = 1000$ r.s. from Pareto($\hat{\beta}$)

Recompute $\hat{\beta}_i^*$ $i = 1, \ldots, B = 1000$

Histogram $\hat{\beta}_i^*$

$\approx$ 95% CI $\left(\hat{\beta}^*_{[1000(.025)]} , \hat{\beta}^*_{[1000(.975)]} \right)$