Stat 6871 Homework 1

(1) $\gamma_X(r,s) = Cov(X_r, X_s) = E[(X_r - \mu_X(r))(X_s - \mu_X(s))]$

$= E[X_r X_s - X_r \mu_X(s) - \mu_X(r) X_s + \mu_X(1)\mu_X(s)]$

$= E[X_r X_s] - E[X_r]\mu_X(s) - \mu_X(r) E[X_s] + \mu_X(r)\mu_X(s)$

$= E[X_r X_s] - \mu_X(r)\mu_X(s) - \mu_X(r)\mu_X(s) + \mu_X(r)\mu_X(s)$

$= E[X_r X_s] - \mu_X(r)\mu_X(s)$ $\square$

(2) Assume X_t is a stationary process and $X_t = \phi X_{t-1} + Z_t$, $t = 0, \pm 1, \ldots$ where Z_t is iid $WN(0, \sigma^2)$, $|\phi| < 1$ and Z_t is uncorrelated with X_s, $s < t$.

<u>mean</u>: $\mu_X = E[X_t] = E[\phi X_{t-1} + Z_t] = \phi E[X_{t-1}] + E[Z_t]$

$= \phi \mu_X + 0$ since X_t is stationary

$\Rightarrow \mu_X = \phi \mu_X \Rightarrow \mu_X = 0$ since $|\phi| < 1$

<u>ACVF</u>: $\gamma_X(h) = Cov(X_{t+h}, X_t) = E[X_{t+h} X_t]$

$= E[(\phi X_{t+h-1} + Z_{t+h}) X_t]$

$= E[\phi X_{t+h-1} X_t] + E[Z_{t+h} X_t]$

$= \phi E[X_{t+h-1} X_t] + E[Z_{t+h}] E[X_t]$

$= \phi \gamma_X(h-1) + 0$ since Z_t & X_s are uncorrelated $t > s$.

So $\gamma_X(h) = \phi \gamma_X(h-1)$

$\Longrightarrow \gamma_X(h) = \phi^h \gamma_X(0)$

Note that $\gamma_X(h) = \gamma(-h)$. so

$\gamma_X(h) = \phi^{|h|} \gamma_X(0) \qquad h = 0, \pm 1, \pm 2, \ldots$

Calculate $\gamma_X(0)$

$\gamma_X(0) = Cov(X_t, X_t) = Cov(X_t, \phi X_{t-1} + Z_t)$

$= \phi Cov(X_t X_{t-1}) + Cov(X_t, Z_t)$

$= \phi \gamma_X(1) + Cov(\phi X_{t-1} + Z_t, Z_t)$

$= \phi \gamma_X(1) + \phi Cov(X_{t-1}, Z_t)$

$+ Cov(Z_t, Z_t)$

note $Cov(X_{t-1}, Z_t) = 0$ since $t-1 < t$.

$\therefore \quad \gamma_X(0) = \phi\, \gamma_X(1) + \sigma^2$.

$\Rightarrow \quad \gamma_X(0) = \phi\, [\phi\, \gamma_X(0)] + \sigma^2$

$\gamma_X(0) = \phi^2\, \gamma_X(0) + \sigma^2$

solving $\quad \gamma_X(0) = \dfrac{\sigma^2}{1-\phi^2}$

$\therefore \quad \gamma_X(h) = \phi^{|h|} \dfrac{\sigma^2}{1-\phi^2} \qquad h = 0, \pm 1, \ldots$

and $\quad \rho_X(h) = \dfrac{\gamma_X(h)}{\gamma_X(0)} = \phi^{|h|} \qquad h = 0, \pm 1, \ldots$

(3) $\quad X_t = Z_{t-1} + 2Z_t + Z_{t+1} \qquad Z_t \overset{iid}{\sim} N(0, \sigma^2)$

mean: $E[X_t] = E[Z_{t-1} + 2Z_t + Z_{t-1}]$

$= E[Z_{t-1}] + 2E[Z_t] + E[Z_{t+1}]$

$= 0$

ACVF: $\quad \gamma_X(h) = Cov(X_{t+h}, X_t) = E[X_{t+h} X_t]$

$= E[(Z_{t+h-1} + 2Z_{t+h} + Z_{t+h+1})(Z_{t-1} + 2Z_t + Z_{t+1})]$

$$= E[Z_{t+h-1}Z_{t-1} + 2Z_{t+h-1}Z_t + Z_{t+h-1}Z_{t+1}$$
$$+ 2Z_{t+h}Z_{t-1} + 4Z_{t+h}Z_t + 2Z_{t+h}Z_{t+1}$$
$$+ Z_{t+h+1}Z_{t-1} + 2Z_{t+h+1}Z_t + Z_{t+h+1}Z_{t+1}]$$

$h=0$ $\quad \gamma_X(0) = E[Z_{t-1}^2 + 4Z_t^2 + Z_{t+1}^2]$

$$= \sigma^2 + 4\sigma^2 + \sigma^2 = 6\sigma^2$$

$h=1$ $\quad \gamma_X(1) = E[2Z_t^2 + 2Z_{t+1}^2] = 4\sigma^2$

$h=-1$ $\quad \gamma_X(-1) = E[2Z_{t-1}^2 + 2Z_t] = 4\sigma^2$

$h=2$ $\quad \gamma_X(2) = E[Z_{t+1}^2] = \sigma^2$

$h=-2$ $\quad \gamma_X(-2) = E[Z_{t-1}^2] = \sigma^2$

$|h| \geq 3$ $\quad \gamma_X(h) = 0$

$$\therefore \quad \gamma_X(h) = \begin{cases} 6\sigma^2 & h=0 \\ 4\sigma^2 & h=\pm 1 \\ \sigma^2 & h=\pm 2 \\ 0 & |h| \geq 3 \end{cases}$$

$$\therefore \quad \rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1 & h=0 \\ 2/3 & h=\pm 1 \\ 1/6 & h=\pm 2 \\ 0 & |h| \geq 3 \end{cases}$$

(4) Problem 1.7 X_t and Y_t are uncorrelated

Show $X_t + Y_t$ is stationary

$$E[X_t + Y_t] = E[X_t] + E[Y_t] = \mu_x + \mu_y.$$

$$\gamma_{X+Y}(h) = E[(X_{t+h} + Y_{t+h} - E[X_{t+h} + Y_{t+h}])$$
$$(X_t + Y_t - E[X_t + Y_t])].$$

$$= E[((X_{t+h} + Y_{t+h}) - (\mu_x + \mu_y))$$
$$((X_t + Y_t) - (\mu_x + \mu_y))].$$

$$= E[(X_{t+h} + Y_{t+h})(X_t + Y_t)]$$
$$- (\mu_x + \mu_y)(\mu_x + \mu_y)$$

$$= E[X_{t+h} X_t + X_{t+h} Y_t + X_t Y_{t+h} + Y_{t+h} Y_t]$$
$$- (\mu_x + \mu_y)^2$$

$$= E[X_{t+h} X_t] + E[X_{t+h} Y_t] + E[X_t Y_{t+h}]$$
$$+ E[Y_{t+h} Y_t] - (\mu_x + \mu_y)^2$$

$$= E[X_{t+h} X_t] - \mu_x^2 + E[Y_{t+h} Y_t] - \mu_y^2$$

$$= E[(X_{t+h} - \mu_x)(\cancel{X_t} - \mu_x)]$$

$$+ E[(Y_{t+h} - \mu_y)(Y_t - \mu_y)]$$

$$= \gamma_X(h) + \gamma_Y(h).$$

$\therefore$ $X_t + Y_t$ is stationary and

$$\gamma_{X+Y}(h) = \gamma_X(h) + \gamma_Y(h).$$

5 $X_t \overset{iid}{\sim} (0, \sigma^2)$ random walk model $S_t = \sum_{s=1}^{t} X_s$.

a) $E[S_t] = E[\sum_{s=1}^{t} X_s] = E[X_1 + X_2 + \cdots + X_t]$.

$$= E[X_1] + E[X_2] + \cdots + E[X_t].$$

$$= 0 + 0 + \cdots + 0 = 0.$$

$$\gamma_S(h) = Cov(S_{t+h}, S_t) = E[S_{t+h} S_t]$$

$$= E[(X_1 + X_2 + \cdots + X_{t+h}) S_t].$$

$$= E[X_1 S_t + X_2 S_t + \cdots + X_{t+h} S_t]$$

$$= E\left[X_1 \sum_{s=1}^{t} X_s\right] + E\left[X_2 \sum_{s=1}^{t} X_s\right] + \cdots$$

$$+ E\left[X_{t+h} \sum_{s=1}^{t} X_s\right].$$

$$= E\left[X_1^2 + X_1 X_2 + \cdots + X_1 X_t\right].$$

note $E[X_1 X_2]=0$ since $X_t \overset{iid}{\sim}(0,\sigma^2)$

$$+ E[X_2^2] + \cdots + E[X_t^2]$$

$$+ 0 + \cdots + 0$$

$$= \underbrace{\sigma^2 + \cdots + \sigma^2}_{t} + \underbrace{0 + \cdots + 0}_{h.}$$

$= t\sigma^2$ nonstationary, $\gamma_X(h)$ depends on t.

b) S_t is nonstationary since $\gamma_X(h) = t\sigma^2$.

c) difference S_t

$$\nabla S_t = S_t - S_{t-1} = \sum_{s=1}^{t} X_s - \sum_{s=1}^{t-1} X_s = X_t$$

$$E[\nabla S_t] = E[X_t] = 0$$

$$\gamma_{\nabla S}(h) = E[X_{t+h} X_t] = \begin{cases} \sigma^2 & h=0 \\ 0 & |h| \geq 1 \end{cases}$$

So $\nabla S_t = X_t$ is stationary.

⑥ $X_t = \beta_0 + \beta_1 t + Z_t \qquad Z_t \overset{iid}{\sim} (0, \sigma^2)$

a) $E[X_t] = E[\beta_0 + \beta_1 t + Z_t] = E[\beta_0 + \beta_1 t] + E[Z_t]$.

$= \beta_0 + \beta_1 t$.

X_t is nonstationary since $E[X_t]$ depends on t.

b) $\nabla X_t = X_t - X_{t-1} = \beta_0 + \beta_1 t + Z_t - (\beta_0 + \beta_1 (t-1) + Z_{t-1})$

$= \cancel{\beta_0} + \cancel{\beta_1 t} + Z_t - \cancel{\beta_0} - \cancel{\beta_1 t} + \beta_1 + Z_t - Z_{t-1}$

$= \beta_1 + Z_t - Z_{t-1}$

$E[\nabla X_t] = E[\beta_1 + Z_t - Z_{t-1}]$

$= \beta_1 + E[Z_t] - E[Z_{t-1}] = \beta_1 + 0 - 0$

$= \beta_1$

$\gamma_{\nabla X}(h) = Cov(\nabla X_{t+h}, \nabla X_t)$

$= E[(\nabla X_{t+h} - E[\nabla X_{t+h}])$

$(\nabla X_t - E[\nabla X_t])]$.

$= E[(\cancel{\beta_1} + Z_{t+h} - Z_{t+h-1} - \cancel{\beta_1})$

$(\cancel{\beta_1} + Z_t - Z_{t-1} - \cancel{\beta_1})]$

$$= E\left[(Z_{t+h} - Z_{t+h-1})(Z_t - Z_{t-1}) \right]$$

$$= E\left[Z_{t+h} Z_t - Z_{t+h} Z_{t-1} - Z_{t+h-1} Z_t + Z_{t+h-1} Z_{t-1} \right]$$

$$\gamma_X(0) = E[Z_t^2] + E[Z_{t-1}^2] = 2\sigma^2$$

$$\gamma_X(1) = -E[Z_t^2] = -\sigma^2$$

$$\gamma_X(-1) = -E[Z_{t-1}^2] = -\sigma^2$$

$$\gamma_X(2) = 0 \qquad \gamma_X(-2) = 0$$

$$\gamma_X(h) = \begin{cases} 2\sigma^2 & h = 0 \\ -\sigma^2 & h = \pm 1 \\ 0 & |h| \geq 2 \end{cases}$$

$$\rho_X(h) = \begin{cases} 1 & h = 0 \\ -\frac{1}{2} & h = \pm 1 \\ 0 & |h| \geq 2 \end{cases}$$