Stat 6871 - Introduction to Time Series Analysis

Homework #2

(1) Recall the ARMA(p,q) model

$$\phi(B) X_t = \theta(B) Z_t$$

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

The summary of the stationary conditions

ARMA(0, q)	always stationary
AR(1) or ARMA(1, q)	$\lvert\phi_1\rvert < 1$
AR(2) or ARMA(2, q)	$\lvert\phi_2\rvert < 1$ and $\phi_2 + \phi_1 < 1$, $\phi_2 - \phi_1 < 1$

Correction: The summary of the invertibility conditions.

ARMA(p, 0)	always invertible
MA(1) or ARMA(p, 1)	$\lvert\theta_1\rvert < 1$
MA(2) or ARMA(p, 2)	$\lvert\theta_2\rvert < 1$ and $\theta_2 + \theta_1 > -1$, $\theta_2 - \theta_1 > -1$

a) $X_t = 0.3 X_{t-1} + Z_t$

$(1 - 0.3) X_t = Z_t$ so $\phi(B) = 1 - 0.3B$

This is an AR(1) model with $\phi_1 = 0.3$, Since $\lvert\phi_1\rvert < 1$ This model is stationary. Or note that the root of $\phi(z) = 0$ is outside the unit circle $1 - 0.3z = 0 \Rightarrow z = \frac{10}{3} > 1$

b) $X_t = Z_t - 1.3Z_{t-1} + 0.4Z_{t-2}$

$X_t = (1 - 1.3B + 0.4B)Z_t$. so $\theta(B) = 1 - 1.3B + 0.4B$

This is a MA(2) model (recall the general form of a MA(2) model

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$$

note the positive signs) with $\theta_1 = -1.3$ and $\theta_2 = 0.4$. Since

$$|\theta_2| = |0.4| = 0.4 < 1 \text{ and}$$

$$\theta_2 + \theta_1 = 0.4 - 1.3 = -.9 > -1 \text{ and}$$

$$\theta_2 - \theta_1 = 0.4 - (-1.3) = 1.7 > -1$$

This model is invertible. Or note that the roots of $\theta(z) = 1$ are outside of the unit circle.

$$1 - 1.3z + 0.4z^2 = 0.$$

roots: $\dfrac{1.3 \pm \sqrt{(1.3)^2 - 4(0.4)(1)}}{2(.4)} = \dfrac{1.3 \pm .3}{.8}$

$$= \frac{1}{.8} = 1.25$$

or

$$= \frac{1.6}{.8} = 2$$

c) $X_t = 0.5 X_{t-1} + Z_t - 1.3 Z_{t-1} + 0.4 Z_{t-2}$

$$(1 - 0.5B) X_t = (1 - 1.3 B + 0.4 B^2) Z_t$$

$$\phi(B) = 1 - 0.5 B$$

$$\theta(B) = 1 - 1.3 B + 0.4 B^2$$

This is an ARMA(1, 2) with $\phi_1 = 0.5$, $\theta_1 = -1.3$, $\theta_2 = 0.4$. The model is stationary since $|\phi_1| < 1$ and it is invertible by b).

a) Write X_t in terms of past values of Z_t, i.e., find the $\{\psi_j\}$ s.t.

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

So

$$X_t = 0.3 X_{t-1} + Z_t$$

$$(1 - 0.3B) X_t = Z_t$$

$$X_t = \frac{1}{1 - 0.3 B} Z_t.$$

recall the geometric series $\frac{1}{1-ax} = \sum_{j=0}^{\infty} (ax)^j$

So $X_t = (1 + 0.3 B + (0.3)^2 B^2 + \cdots) Z_t$

$$= Z_t + 0.3 Z_{t-1} + (0.3)^2 Z_{t-2} + \cdots$$

$$= \sum_{j=0}^{\infty} (0.3)^j Z_{t-j} \qquad \psi_j = (0.3)^j$$

(2) Find $\{\psi_j\}$.

$$(1-.5B)X_t = (1+.4B)Z_t$$

$$X_t = (1+.4B)(1-.5B)^{-1}Z_t$$

$$= (1+.4B)(1+.5B+.5^2B^2+\cdots)Z_t$$

$$= (1+(.9)B + (.9)(.5)B^2 + (.9)(.5)^2B^3+\cdots)Z_t$$

So $\psi_j = (.9)(.5)^{j-1} \quad j=1,2,\ldots$

Find $\{\pi_j\}$.

$$(1-.5B)X_t = (1+.4B)Z_t$$

$$(1-.5B)(1+.4B)^{-1}X_t = Z_t.$$

$$(1-.5B)(1-.4B+.4^2B^2-.4^3B^3+\cdots)X_t = Z_t$$

$$(1-.9B+(.9)(.4)B^2-(.9)(.4)^2B^3+\cdots)X_t = Z_t.$$

So $\pi_j = (-1)^j(.9)(.4)^{j-1} \quad j=1,2,\ldots$

(3) a) $(1-B)(1-0.2B)X_t = (1-0.5B)Z_t$

This is an ARIMA(1, 1, 1) model.

with $\phi = 0.2 \quad \theta = -0.5$

b) This is a stationary process since $|\phi| < 1$

c) $X_t = \psi(B) Z_t \qquad \psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$

$$\psi(B) = \frac{\theta(B)}{\phi(B)} = \frac{1 - .5B}{(1-B)(1-0.2B)} = \frac{1 - .5B}{1 - 1.2B + 0.2B^2}$$

Using polynomial division

$$
\begin{array}{r l}
 & 1 + .7B + .64B^2 + .628B^3 \\
1 - 1.2B + .2B^2 & \overline{)\,1 - .5B} \\
 & \underline{\ominus 1 \oplus 1.2B \ominus + .2B^2} \\
 & .7B - .2B^2 \\
 & \underline{\ominus .7B \oplus - .84B^2 \ominus + .14B^2} \\
 & .64B^2 - .14B^3 \\
 & \underline{\ominus .64B^2 \oplus - .768B^3 \ominus + .128B^4} \\
 & .628B^3 - .128B^4
\end{array}
$$

So $\psi_1 = .7$, $\psi_2 = .64$, $\psi_3 = .628$

d) $Z_t = \pi(B) X_t \qquad \pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots$

$$\pi(B) = \frac{\phi(B)}{\theta(B)} = \frac{(1-B)(1-0.2B)}{1-.5B} = \frac{1-1.2B+0.2B^2}{1-.5B}$$

$$\begin{array}{r l}
 & 1 - .7B - .15B^2 - .075B^3 - .0375B^4 \\
1-.5B & \overline{)\,1 - 1.2B + .2B^2} \\
 & \ominus 1 \ominus .5B \\
 & \overline{\quad -.7B + .2B^2} \\
 & \oplus -.7B \ominus +.35B^2 \\
 & \overline{\qquad -.15B^2} \\
 & \oplus -.15B^2 \ominus +.075B^2 \\
 & \overline{\qquad -.075B^3} \\
 & \oplus -.075B^3 \ominus +.0375B^4 \\
 & \overline{\qquad -.0375B^4}
\end{array}$$

$\pi_1 = -.7, \quad \pi_2 = -.15, \quad \pi_3 = -.075, \quad \pi_4 = -.0375$

— (4) Problem 2.2

$X_t = A\cos(wt) + B\sin(wt) \qquad t = 0, \pm 1, \ldots$

A and B are uncorrelated.

$E[A] = 0$, $Var(A) = 1$, $E[B] = 0$, $Var(B) = 1$

$w \in [0, \pi]$.

Show that this process is stationary

$$E[X_t] = E[A\cos(wt) + B\sin(wt)].$$

$$= E[A]\cdot\cos(wt) + E[B]\cdot\sin(wt)$$

$$= 0 + 0 = \underline{\underline{0}}$$

$$\gamma_X(h) = E[X_{t+h} X_t]$$

$$= E[\{A\cos[w(t+h)] + B\sin[w(t+h)]\} \cdot \{A\cos(wt) + B\sin(wt)\}]$$

$$= E[A^2\cos[w(t+h)]\cos(wt)$$
$$+ AB\cos[w(t+h)]\sin(wt)$$
$$+ AB\sin[w(t+h)]\cos(wt)$$
$$+ B^2\sin[w(t+h)]\sin(wt)]$$

note: $\sin(U+V) = \sin U \cos V + \cos U \sin V$

so

$$= E\big[\, A^2 \cos[w(t+h)] \cos(wt) + AB \sin[w(t+h) + wt] + B^2 \sin[w(t+h)] \sin(wt) \,\big]$$

$$= E[A^2] \cdot \cos[w(t+h)] \cos(wt) + E[A]E[B] \sin[w(t+h) + wt] + E[B^2] \sin[w(t+h)] \sin(wt)$$

$$= \sigma^2 \big[\cos[w(t+h)] \cos(wt) + \sin[w(t+h)] \sin(wt) \big]$$

note: $\cos(U-V) = \cos U \cos V + \sin U \sin V$.

So

$$= \sigma^2 \cos[w(t+h) - wt]$$

$$= \sigma^2 \cos(wh).$$

$\therefore \quad \gamma_X(h) = \sigma^2 \cos(wh) \qquad h = 0, \pm 1, \ldots$

Since $E[X_t]$ is constant and $\gamma_X(h)$ only depends on the lag h, X_t is stationary.

$K(h) = \cos(wh)$ is nonnegative definite since $\gamma_X(h)$ is an ACVF! see Remark 2 p 46

(5) Problem 2.3

a) Find The ACVF of $X_t = Z_t + .3 Z_{t-1} - .4 Z_{t-2}$

$E[X_t] = 0$

$\gamma_X(h) = E[X_{t+h} X_t]$

$= E[(Z_{t+h} + .3 Z_{t+h-1} - .4 Z_{t+h-2})$
$(Z_t + .3 Z_{t-1} - .4 Z_{t-2})]$

$= E[Z_{t+h} Z_t + .3 Z_{t+h} Z_{t-1} - .4 Z_{t+h} Z_{t-2}$
$+ .3 Z_{t+h-1} Z_t + .09 Z_{t+h-1} Z_{t-1} - .12 Z_{t+h-1} Z_{t-2}$
$- .4 Z_{t+h-2} Z_t - .12 Z_{t+h-2} Z_{t-1} + .16 Z_{t+h-2} Z_{t-2}]$

$h=0 \quad \gamma_X(0) = E[Z_t^2] + .09 E[Z_{t+1}^2] + .16 E[Z_{t-2}^2]$

$= 1.25 \sigma^2.$

$h=1 \quad \gamma_X(1) = .3 E[Z_t^2] - .12 E[Z_{t-1}^2] = .18 \sigma^2$

$h=2 \quad \gamma_X(2) = -.4 E[Z_t^2] = -.4 \sigma^2$

$h \geq 3 \quad \gamma_X(h) = 0$

$$\therefore \gamma_X(h) = \begin{cases} 1.25\sigma^2 & h=0 \\ .18\sigma^2 & h=\pm 1 \\ -.4\sigma^2 & h=\pm 2 \\ 0 & |h| \geq 3 \end{cases}$$

b) Find the ACVF of $Y_t = \tilde{Z}_t - 1.2\tilde{Z}_{t-1} - 1.6\tilde{Z}_{t-2}$

$E[Y_t] = 0$

$$\gamma_Y(h) = E[Y_{t+h} Y_t]$$

$$= E[(Z_{t+h} - 1.2 Z_{t+h-1} - 1.6 Z_{t+h-2}) \cdot (Z_t - 1.2 Z_{t-1} - 1.6 Z_{t-2})]$$

$$= E[Z_{t+h} Z_t - 1.2 Z_{t+h} Z_{t-1} - 1.6 Z_{t+h} Z_{t-2} - 1.2 Z_{t+h-1} Z_t + 1.44 Z_{t+h-1} Z_{t-1} + 1.92 Z_{t+h-1} Z_{t-2} - 1.6 Z_{t+h-2} Z_t + 1.92 Z_{t+h-2} Z_{t-1} + 2.56 Z_{t+h-2} Z_{t-2}]$$

$h=0$ $\quad \gamma_Y(0) = E[Z_t^2] + 1.44 E[Z_{t-1}^2] + 2.56 E[Z_{t-2}^2]$

$$= 5\sigma^2$$

$h=1$ $\quad \gamma_Y(1) = -1.2 E[Z_t^2] + 1.92 E[Z_{t-1}^2] = .72\sigma^2$

$h=2$ $\quad \gamma_Y(2) = -1.6 E[Z_t^2] = -1.6\sigma^2$

$h \geq 3$ $\quad \gamma_Y(h) = 0$

$$\therefore \quad \gamma_Y(h) = \begin{cases} 5\sigma^2 & h=0 \\ .72\sigma^2 & h=\pm 1 \\ -1.6\sigma^2 & h = \pm 2 \\ 0 & |h| \geq 3 \end{cases}$$

Note: $\gamma_X(h) = \frac{1}{4}\gamma_Y(h)$.

Question: Why is this? One model is not invertible. Y_t.

(6) Problem 2.9 This is what is called as signal-plus-noise model.

$Y_t = X_t + W_t \qquad W_t \sim WN(0, \sigma_W^2)$

X_t is an AR(1) processes.

$X_t - \phi X_{t-1} = Z_t \qquad Z_t \sim WN(0, \sigma_Z^2)$

and $E[W_s Z_t] = 0 \quad \forall\, s, t.$

a) Show Y_t is stationary.

$E[Y_t] = E[X_t + W_t] = E[X_t] + E[W_t]$

$= 0 + 0 = 0$

recall for X_t AR(1) $E[X_t] = 0$, HW #1.

$\gamma_h(h) = E[Y_{t+h} Y_t] = E[(X_{t+h} + W_{t+h})(X_t + W_t)]$.

$= E[X_{t+h} X_t + X_{t+h} W_t + W_{t+h} X_t + W_{t+h} W_t]$.

$= E[X_{t+h} X_t] + E[W_{t+h} W_t]$ since $E[W_s Z_t] = 0$

$= \gamma_X(h) + \gamma_W(h)$.

note $E[X_{t+h} W_t] = \phi E[X_{t+h-1} W_t] + 0$

$\Longrightarrow E[X_{t+h-1} W_t] = 0$ since $\phi \neq 1$.

b) $U_t = Y_t - \phi Y_{t-1} = X_t + W_t - \phi X_{t-1} - \phi W_{t-1}$

$= X_t - \phi X_{t-1} + W_t - \phi W_{t-1}$

$= W_t - \phi W_{t-1} + Z_t.$

$E[U_t] = 0$

$\gamma_u(h) = E[U_{t+h} U_t]$

$= E[(W_{t+h} - \phi W_{t+h-1} + Z_{t+h})(W_t - \phi W_{t-1} + Z_t)].$

$= E[W_{t+h} W_t - \phi W_{t+h} W_{t-1} + W_{t+h} Z_t$

$- \phi W_{t+h-1} W_t + \phi^2 W_{t+h-1} W_{t-1} - \phi W_{t+h-1} Z_t$

$+ Z_{t+h} W_t - \phi Z_{t+h} W_{t-1} + Z_{t+h} Z_t].$

$h=0 \quad \gamma_u(0) = E[W_t^2 + \phi^2 W_{t-1}^2 + Z_t^2].$

$= (1+\phi^2)\sigma_w^2 + \sigma_z^2.$

$h=1 \quad \gamma_u(1) = E[-\phi W_t^2] = -\phi \sigma_w^2$

$$\gamma_u(h) = \begin{cases} (1+\phi^2)\sigma_w^2 + \sigma_z^2 & h = 0 \\ -\phi \sigma_w^2 & h = \pm 1 \\ 0 & |h| \geq 2 \end{cases}$$

Since U_t is 1-correlated by Prop 2.1.1 it is a MA(1) process, ie, we can write

$$U_t = Z_t + \theta Z_{t-1} \qquad Z_t \sim WN(0, \sigma_z^2)$$

so

$$\gamma_U(h) = \begin{cases} (1+\theta^2)\sigma_z^2 & h = 0 \\ \theta \sigma_z^2 & h = \pm 1 \\ 0 & |h| \geq 1 \end{cases}$$

see p. 16

equating corresponding the ACVF's

$$(1+\theta^2)\sigma_z^2 = (1+\phi^2)\sigma_w^2 + \sigma_z^2$$

$$(1+\theta^2)\sigma_z^2 - \sigma_z^2 = (1+\phi^2)\sigma_w^2$$

$$\theta^2 \sigma_z^2 = (1+\phi^2)\sigma_w^2$$

$$\theta^2\left[-\frac{\phi}{\theta}\sigma_w^2\right] = (1+\phi^2)\sigma_w^2$$

$$\boxed{\theta = -\frac{(1+\phi^2)}{\phi}}$$

$$\theta \sigma_z^2 = -\phi \sigma_w^2$$

$$\boxed{\sigma_z^2 = -\frac{\phi}{\theta}\sigma_w^2}$$

So

$$Y_t - \phi Y_t = Z_t + \theta Z_{t-1} \qquad Z_t \sim WN(0, \sigma_Z^2)$$

and

ϕ is the AR coefficient.

$\theta = -\left(\frac{1+\phi^2}{\phi}\right)$ is the MA coefficient

$\sigma_Z^2 = \left(-\frac{\phi}{\theta}\right)\sigma_W^2 = \left(\frac{\phi^2}{1+\phi^2}\right)\sigma_W^2$ is the variance of the WN.

(6) Problem 2.16

Sunspots.txt Plot 1

ACF and PACF of D_t and X_t Plot 2.

AR(2) model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t \quad Z_t \sim WN(0, \sigma^2)$$

Estimated parameters

$$\hat{\phi}_1 = 1.3175 \quad \hat{\phi}_2 = -0.6341 \quad \hat{\sigma}^2 = 298.1588$$

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### Homework #2
### Solution to Problem 6

###########################################################################################

Dt <- rts(matrix(scan("I:\\Courswrk\\Stat\\esuess\\Stat6871\\Data_bd\\Sunspots.txt"), ncol=1,
   byrow=T))

ts.plot(Dt, main = "Sunspot data")

# plot the acf up to lag 40

par(mfrow=c(2,2))

acf(Dt)

acf(Dt, type="partial")

# mean correct the data.

Xt <- Dt - mean(Dt)

acf(Xt)

acf(Xt, type="partial")

# fit the AR(2) using the Yule-Walker option

Xt.ar2.yw <- ar.yw(Xt, order=2)

Xt.ar2.yw

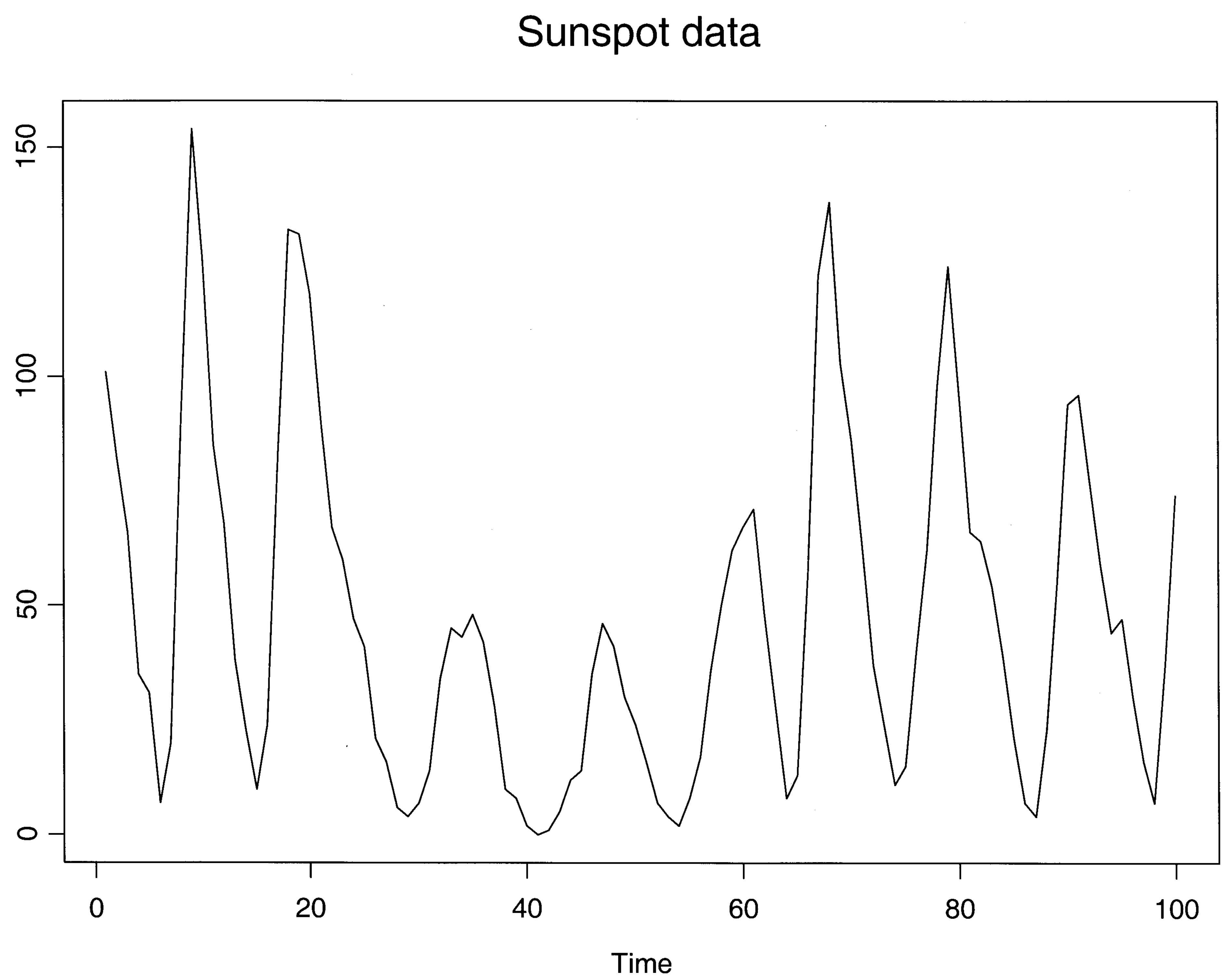
# look at the residuals

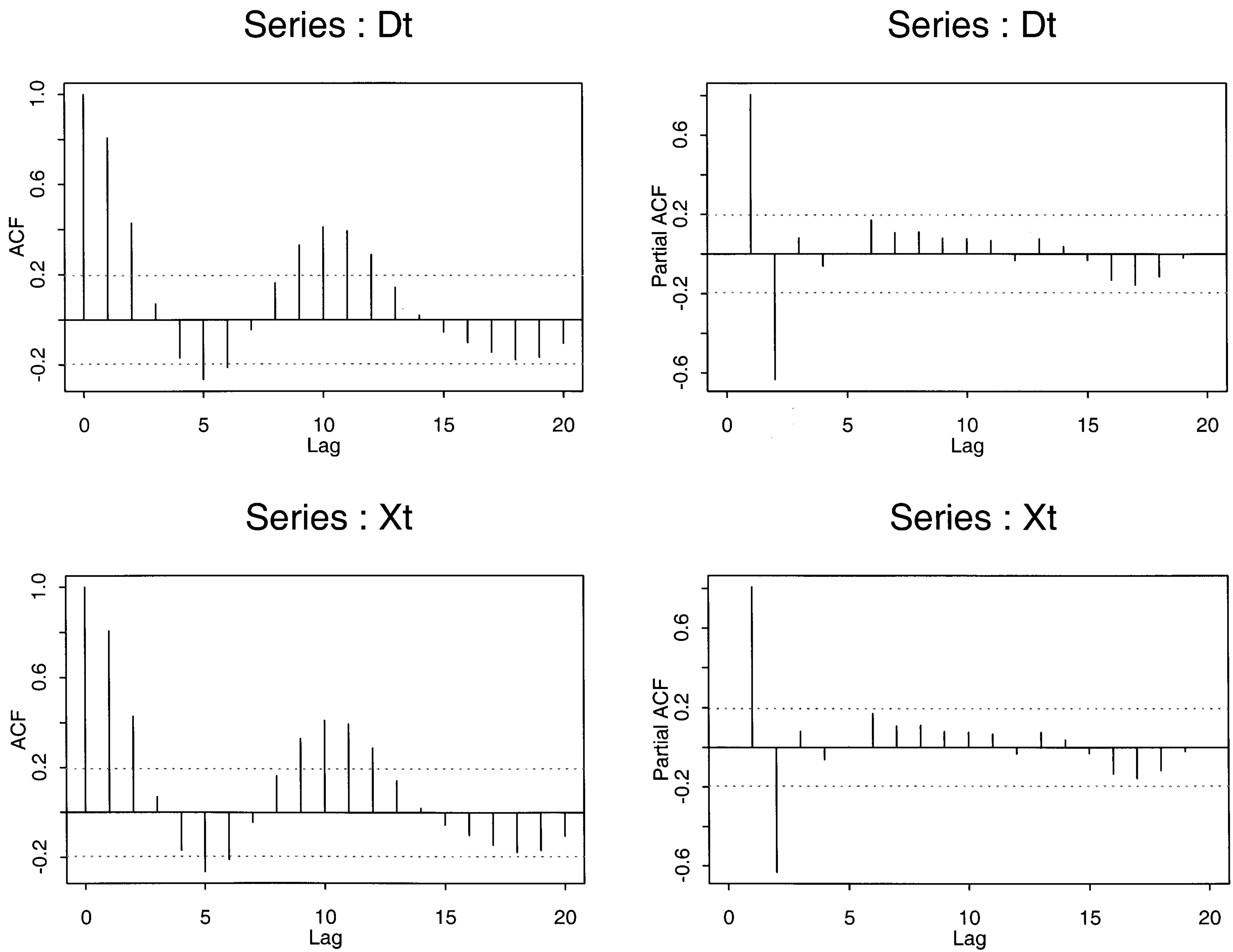
Rt <- Xt.ar2.yw$resid

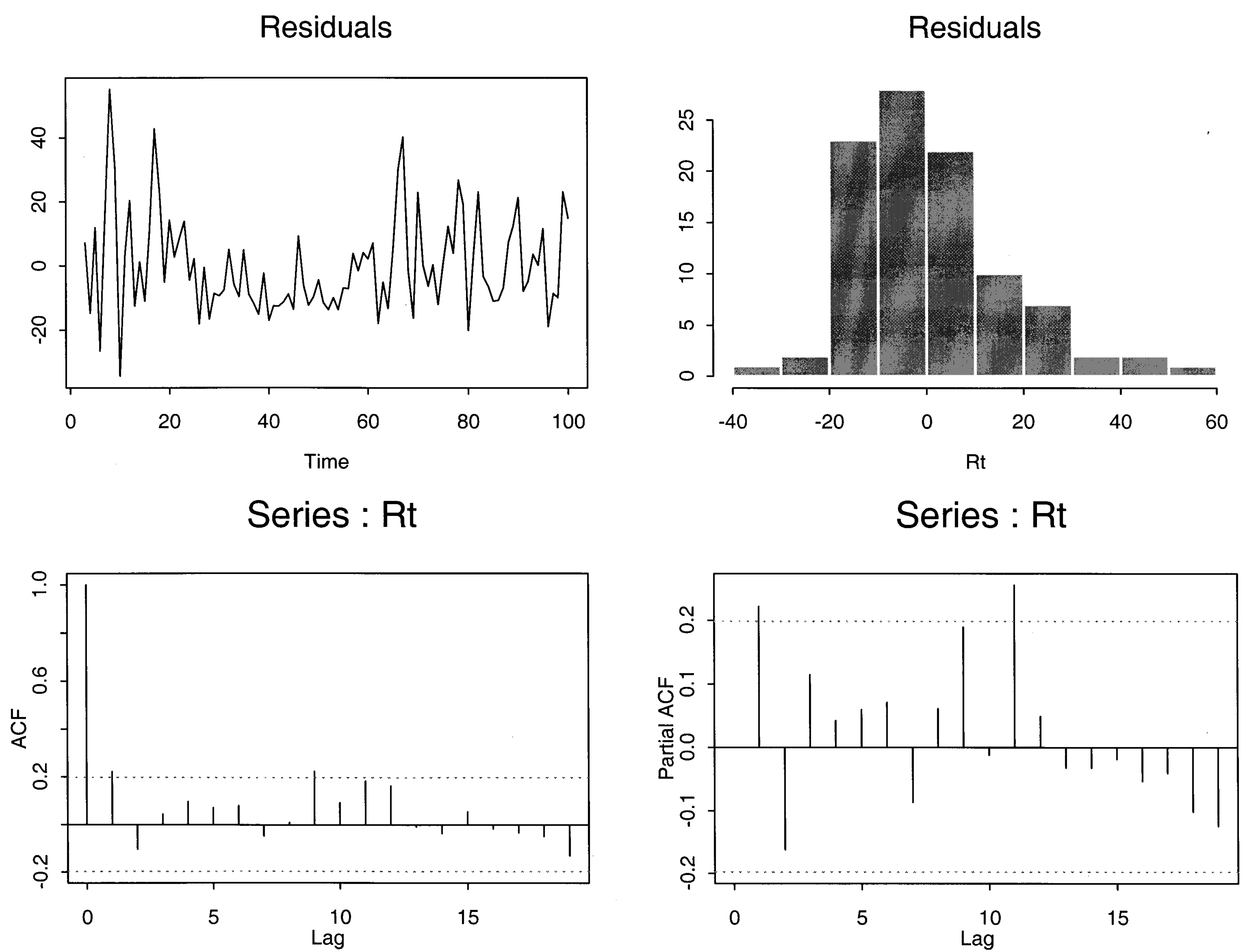
ts.plot(Rt, main="Residuals")

hist(Rt, main = "Residuals")

acf(Rt)
acf(Rt, type="partial")
```


Sunspot data
150
100
50
0
0
20
40
60
80
100
Time


Series : Dt
ACF
Lag
Series : Dt
Partial ACF
Lag
Series : Xt
ACF
Lag
Series : Xt
Partial ACF
Lag


Residuals
-20
0
20
40
0
20
40
60
80
100
Time
Residuals
0
5
10
15
20
25
-40
-20
0
20
40
60
Rt
Series : Rt
ACF
-0.2
0.2
0.6
1.0
0
5
10
15
Lag
Series : Rt
Partial ACF
-0.2
-0.1
0.0
0.1
0.2
0
5
10
15
Lag