Homework 3

(1) $E[(X-\mu_X)(Y-\mu_Y)]$

$$= E[(X-\mu_X)Y - (X-\mu_X)\mu_Y]$$

$$= E[(X-\mu_X)Y] - E[(X-\mu_X)\mu_Y]$$

$$= E[(X-\mu_X)Y] - \{\mu_X\mu_Y - \mu_X\mu_Y\}$$

$$= E[(X-\mu_X)Y]$$

(2) Problem 1.1.

a) $E[(Y-c)^2] = E[Y^2 - 2Yc + c^2]$

$$= E[Y^2] - 2E[Y]c + c^2$$

$$\frac{d}{dc} E[(Y-c)^2] = -2E[Y] + 2c$$

Minimize by setting the derivative equal to zero

$$-2E[Y] + 2c = 0$$

$$2c = 2E[Y]$$

$$c = E[Y].$$

$$\Rightarrow c = \mu.$$

b.) $E[(Y-f(X))^2 | X]$

$$= E[Y^2 - 2Y f(X) + f^2(X) | X]$$

$$= E[Y^2 | X] - 2E[Y|X] \cdot f(X) + f^2(X)$$

$$\frac{d}{d f(X)} E[(Y-f(X))^2 | X]$$

$$= -2E[Y|X] + 2f(X)$$

setting equal to zero

$$f(X) = E[Y|X].$$

c) $E[(Y-f(X))^2] = E[E[(Y-f(X))^2 | X]]$.

since the inside expectation is minimized.
The uncondition expectation is minimized

(3.) Determine a_0 & a_1

$$S(a_0, a_1) = E[(Y - a_0 - a_1 X)^2].$$

$$= E[Y^2 - 2Ya_0 - 2Ya_1 X + 2a_0 a_1 X + a_0^2 + a_1^2 X]$$

$$= E[Y^2] - 2E[Y]a_0 - 2E[YX]a_1$$

$$+ 2a_0 a_1 E[X] + a_0^2 + a_1^2 E[X^2].$$

find the minimum,

$$\frac{\partial}{\partial a_0} S(a_0, a_1) = -2E[Y] + 2a_1 E[X] + 2a_0 = 0$$

$$\frac{\partial}{\partial a_1} S(a_0, a_1) = -2E[YX] + 2a_0 E[X] + 2a_1 E[X^2] = 0$$

two equations in two unknowns, solve for a_0 & a_1

$$\hat{a}_1 = \frac{E[XY] - E[X]E[Y]}{E[X^2] - (E[X])^2}$$

$$= \frac{E[(X - E[X])(Y - E[Y])]}{E[(X - E[X])^2]} = \frac{\sigma_{XY}}{\sigma_X^2}$$

$$\hat{a}_0 = E[Y] - \hat{a}_1 E[X].$$

$$MSPE = E[(Y - \hat{a}_0 - \hat{a}_1 X)^2]$$

$$= E[(Y - E[Y] + \hat{a}_1 E[X] - \hat{a}_1 X)^2]$$

$$= E[\{(Y - E[Y]) - \hat{a}_1 (X - E[X])\}^2]$$

$$= E[(Y - E[Y])^2] + \hat{a}_1^2 E[(X - E[X])^2]$$

$$- 2\hat{a}_1 E[(Y - E[Y])(X - E[X])]$$

$$= \sigma_Y^2 + \hat{a}_1^2 \sigma_X^2 - 2\hat{a}_1 \sigma_{XY}$$

$$= \sigma_Y^2 + \frac{\sigma_{XY}^2}{\sigma_X^2} - 2\frac{\sigma_{XY}^2}{\sigma_X^2}$$

$$= \sigma_Y^2 - \frac{\sigma_{XY}^2}{\sigma_X^2}$$

$$= \sigma_Y^2 - \rho^2 \sigma_Y^2 \qquad \rho^2 = \frac{\sigma_{XY}^2}{\sigma_X^2 \sigma_Y^2} = \left(\frac{\sigma_{XY}}{\sqrt{\sigma_X^2}\sqrt{\sigma_Y^2}}\right)^2$$

(4) $\phi(B)\,\Phi(B)\,\nabla^d \nabla_s^D X_t = \theta(B)\,\Theta(B)\,Z_t$

$Z_t \sim WN(0, \sigma^2)$.

SARIMA $(0,0,1)(1,1,0)_{12}$

$(1-\Phi_1 B^{12})(1-B^{12})X_t = (1-\theta_1 B)Z_t$

$(1-\Phi_1 B^{12})(X_t - X_{t-12}) = Z_t - \theta_1 Z_{t-1}$

$X_t - X_{t-12} - \Phi_1 X_{t-12} + \Phi_1 X_{t-24} = Z_t - \theta_1 Z_{t-1}$

model: $X_t = X_{t-12} + \Phi_1 X_{t-12} - \Phi_1 X_{t-24} + Z_t - \theta_1 Z_{t-1}$

$$
\begin{aligned}
P_n X_{n+1} &= P(X_{n+1} \mid \underset{\sim}{X}) \\
&= P(X_{n-11} + \Phi_1 X_{n-11} - \Phi_1 X_{n-23} + Z_{n+1} - \theta_1 Z_n \mid \underset{\sim}{X}) \\
&= X_{n-11} + \Phi_1 X_{n-11} - \Phi_1 X_{n-23} + P(Z_{t+1} \mid \underset{\sim}{X}) \\
&\qquad - \theta_1 P(Z_n \mid \underset{\sim}{X}) \\
&= X_{n-11} + \Phi_1 X_{n-11} - \Phi X_{n-23} - \theta_1 Z_n .
\end{aligned}
$$

where $Z_n = X_n - \hat{X}_n$

$\hat{X}_n = P_n X_n$

$$P_n X_{n+2} = X_{n-10} + \Phi X_{n-10} - \Phi X_{n-22} - \theta_1 P(Z_{n+1} \mid \underset{\sim}{X})$$

$$= X_{n-10} + \Phi X_{n-10} - \Phi X_{n-22}$$

$$P_n X_{n+3} = X_{n-9} + \Phi X_{n-9} - \Phi X_{n-21}$$

$$P_n X_{n+4} = X_{n-8} + \Phi X_{n-8} - \Phi X_{n-20}$$

$$P_n X_{n+5} = X_{n-7} + \Phi X_{n-7} - \Phi X_{n-19}$$

$$P_n X_{n+6} = X_{n-6} + \Phi X_{n-6} - \Phi X_{n-18}$$

$$P_n X_{n+7} = X_{n-5} + \Phi X_{n-5} - \Phi X_{n-17}$$

$$P_n X_{n+8} = X_{n-4} + \Phi X_{n-4} - \Phi X_{n-16}$$

$$P_n X_{n+9} = X_{n-3} + \Phi X_{n-3} - \Phi X_{n-15}$$

$$P_n X_{n+10} = X_{n-2} + \Phi X_{n-2} - \Phi X_{n-14}$$

$$P_n X_{n+11} = X_{n-1} + \Phi X_{n-1} - \Phi X_{n-13}$$

$$P_n X_{n+12} = X_n + \Phi X_n - \Phi X_{n-12}$$

$$P_n X_{n+13} = P_n X_{n+1} + \Phi P_n X_{n+1} - \Phi X_{n-11}$$

(3) $(1-B)(1-0.2B)X_t = (1-0.5B)Z_t$

$(1-1.2B+0.2B^2)X_t = (1-0.5B)Z_t$

$X_t - 1.2X_{t-1} + 0.2X_{t-2} = Z_t - 0.5Z_{t-1}$

$X_t = 1.2X_{t-1} - 0.2X_{t-2} + Z_t - 0.5Z_{t-1}$

$P_n X_{n+1} = 1.2X_n - 0.2X_{n-1} + P(Z_{n+1}|\underset{\sim}{X}) - 0.5P(Z_n|\underset{\sim}{X})$

$= 1.2X_n - 0.2X_{n-1} - 0.5 P_n Z_n.$

$P_n Z_n = X_n - P_n X_n.$

$P_n X_{n+2} = 1.2P_n X_{n+1} - 0.2X_n - 0.5P(Z_{n+1}|\underset{\sim}{X})$

$= 1.2P_n X_{n+1} - 0.2X_n.$

$P_n X_{n+k} = 1.2P_n X_{n+(k-1)} - 0.2P_n X_{n+(k-2)}.$ ✓

(6) Theoretical PACF of AR(p)

$$\alpha(p) = \phi_p$$

$$\alpha(h) = 0 \quad h > p.$$

Theoretical PACF of MA(q).

$$\alpha(h) = \phi_{hh} = -\frac{(-\theta)^h}{1+\theta^2+\cdots+\theta^{2h}}.$$