Instantaneous Growth Rates

Since taking the difference in logs is a standard technique to achieve stationarity for many economic time series, it is useful to how the difference in logs relates to growth rates. For many series, it is in fact the rate of return that is of primary interest. Stock price levels, for example, lack intrinsic interest for the most part, although the expected return or change from any given price level is a subject of considerable interest.

Let's begin with compounding in discrete time:

$$X_{t+1} = (1+r)X_t$$
$$X_{t+2} = (1+r)X_{t+1} = (1+r)(1+r)X_t$$

where r is the annual interest rate. Now, instead of annual compounding as shown above, we compound interest in m sub-annual periods. Then,

$$X_{t+1} = \left(1+\frac{r}{m}\right)^{mt} X_t$$

represents the rate of growth where compounding occurs in each m period. So, for example, if compounding occurs monthly, then

$$X_{t+1} = \left(1+\frac{r}{12}\right)^{12t} X_t$$

If you start with an amount $X(0)$ and want to know what X will grow to at the end of three years, then by backward substitution,

$$X_3 = \left(1+\frac{r}{12}\right)^{12} X_2 = \left(1+\frac{r}{12}\right)^{12}\left(1+\frac{r}{12}\right)^{12} X_1 = \left(1+\frac{r}{12}\right)^{12}\left(1+\frac{r}{12}\right)^{12}\left(1+\frac{r}{12}\right)^{12} X_0 = \left(1+\frac{r}{12}\right)^{36} X_0$$

Note this is saying that the per-period interest rate is $r/12$ and the principal plus interest is being compounded 36 times.

Next, the intermediate objective is analyzing the effect of increasing the number of compounding periods to an infinite number. First, redefine the basic equation above as a function of m, using a fixed initial amount X_0.

$$V(m) = \left(1 + \frac{r}{m}\right)^{mt} X_0$$

$$V(m) = \left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{rt} X_0$$

$$V(m) = \left[\left(1 + \frac{1}{w}\right)^{w}\right]^{rt} X_0 \text{ where } w = \frac{m}{r}.$$

Now define $V \equiv \lim_{m,w\to\infty} V(m) = \lim_{m,w\to\infty} \left[\left(1 + \frac{1}{w}\right)^{w}\right]^{rt} X_0$.

Since e is defined as $\lim_{w\to\infty} \left(1 + \frac{1}{w}\right)^{w} \equiv e$, by substitution we have

$$V = e^{rt} X_0$$

$$\ln V = rt + \ln X_0$$

$$\ln V - \ln X_0 = rt$$

$$r = \frac{\ln V - \ln X_0}{t} = \frac{d \ln V}{dt} = \frac{\frac{dV}{dt}}{V}$$

So, the difference in logs is the product of the growth rate per period times the number of periods in question. A common method of computing successive one-period returns is differencing logs, in lieu of computing percentage changes. Note that the expression $\frac{\frac{dV}{dt}}{V}$ is intuitively the continuous-time analog of the discrete-time percentage change computation. Also note that the result for r given above is equivalent to the approximation in Shumway (2000) on page 145 in the special case of a single period return where t=1.

Reference: Chiang, Alpha C. (1984). Fundamental Methods of Mathematical Economics, 3rd ed. pp. 274-9.