Homework 3

Example 8.1.1 election polling.

prior $\pi \sim \mathrm{Beta}(a,b)$ $\quad p(\pi) = \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \pi^{a-1}(1-\pi)^{b-1}$

p. 93 $a = 330$
$b = 270$

likelihood $X_1, \dots, X_n \overset{iid}{\sim} \mathrm{Ber}(\pi)$

$X = \sum^n X_i \sim \mathrm{Bin}(n, \pi)$

$p(x|\pi) = \binom{n}{x} \pi^x (1-\pi)^{n-x}$

p. 96 data
$x = 620$
$n = 1000$

posterior $\quad p(\pi|x) = \dfrac{p(\pi)\, p(x|\pi)}{\int p(\pi)\, p(x|\pi)\, d\pi}$.

$$p(\pi|x) \propto p(\pi)\, p(x|\pi)$$
$$\propto \pi^{a-1}(1-\pi)^{b-1} \pi^x (1-\pi)^{n-x}$$
$$\propto \pi^{a+x-1}(1-\pi)^{b+n-x-1}$$

$\therefore \ \pi|x \sim \mathrm{Beta}(a+x,\ b+n-x)$

posterior mean $\quad E[\pi|x] = \int_0^1 \pi\, p(\pi|x)\, d\pi$

$$= \frac{a+x}{a+b+n}$$

$$= \left(\frac{a}{a+b}\right)\left[\frac{a+b}{a+b+n}\right] + \left(\frac{x}{n}\right)\left[\frac{n}{a+b+n}\right]$$

p. 96 $\quad \widehat{E[\pi|x]} = \dfrac{330+620}{330+270+1000} = \dfrac{950}{1600} = .59375$

Example 8.1.1 Beta-Binomial Election polling.

data

```
list(X = 620, n = 1000)          # frequentist estimate 0.620
```

inits

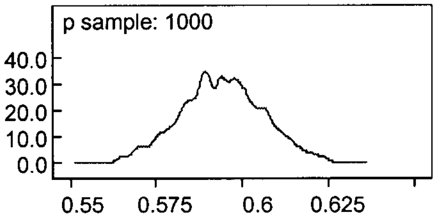
```
list(p = 0.25)                   # starting values for p
```

model

```
model;
{
  X ~ dbin(p,n)
  p ~ dbeta(330,270)             # prior mode = 0.55, 95% prior interval (0.51, 0.59)
}
```

Output:

posterior density (for p)



node	mean	sd	MC error	2.5%	median	97.5%	start	sample
p	0.5941	0.0123	4.757E-4	0.5699	0.5939	0.6182	1001	1000

Bayesian estimate of p

Bayesian 95% credible interval for p.

Example 8.1.2 Weighing an object.

prior $\mu \sim N(\mu_0, \sigma_0^2)$ p 93 $\mu_0 = 200$, $\sigma_0 = 10$

$$p(\mu) \propto \exp\left[-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right].$$

likelihood $x_1, \ldots, x_n \overset{iid}{\sim} N(\mu, \sigma^2)$ $\sigma^2 = 1$ known.

$$p(x_1, \ldots, x_n / \mu) \propto \exp\left[-\frac{1}{2\sigma^2}\sum^n (x_i - \mu)^2\right].$$

p. 97 data $n=5$ $\bar{x} = 197.79$

posterior $p(\mu / x_1, \ldots, x_n) \propto p(\mu) p(x_1, \ldots, x_n / \mu)$

$$\propto \exp\left[-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2\right] \cdot \exp\left[-\frac{1}{2\sigma^2}\sum^n (x_i-\mu)^2\right].$$

$$\propto \exp\left[-\frac{1}{2}\left\{\frac{(\mu-\mu_0)^2}{\sigma_0^2} + \sum^n \frac{(x_i-\mu)^2}{\sigma^2}\right\}\right].$$

complete the square

$$\propto \exp\left[-\frac{1}{2}\left(\frac{\mu^2}{\sigma_0^2} - \frac{2\mu\mu_0}{\sigma_0^2} + \frac{\mu_0^2}{\sigma_0^2} + \frac{\Sigma x_i^2}{\sigma^2} - \frac{2\mu\Sigma x_i}{\sigma^2} + \frac{n\mu^2}{\sigma^2}\right)\right]$$

$$\propto \exp\left[-\frac{1}{2}\left(\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)\mu^2 - 2\left(\frac{\mu_0}{\sigma_0^2} + \frac{\Sigma x_i}{\sigma^2}\right)\mu + \underbrace{\left(\frac{\mu_0^2}{\sigma_0^2} + \frac{\Sigma x_i^2}{\sigma^2}\right)}_{\text{constant.}}\right)\right]$$

$$\propto \exp\left[-\frac{1}{2}\left(\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)\mu^2 - 2\left(\frac{\mu_0}{\sigma_0^2} + \frac{\Sigma x_i}{\sigma^2}\right)\mu\right)\right]$$

$$\propto \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)\left(\mu^2 - 2\frac{\left(\frac{\mu_0}{\sigma_0^2} + \frac{\Sigma x_i}{\sigma^2}\right)}{\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)}\mu + k - k\right)\right]$$

$$\text{let} \quad k = \left[\frac{\left(\frac{\mu_0}{\sigma_0^2} + \frac{\Sigma x_i}{\sigma^2}\right)}{\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)}\right]^2$$

to complete the square add a constant, the $-k$ goes away with $\propto$

$$\propto \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)\left(\mu-\frac{\left(\frac{\mu_0}{\sigma_0^2}+\frac{\Sigma x_i}{\sigma^2}\right)}{\left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)}\right)^2\right].$$

$$\therefore\ \mu \mid x_1, \dots, x_n \sim N(\mu_n, \sigma_n^2)$$

posterior mean. $\mu_n = \dfrac{\left(\frac{\mu_0}{\sigma_0^2}+\frac{n\bar{x}}{\sigma^2}\right)}{\left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)} = \mu_0\left(\sigma_n^2 \cdot \frac{1}{\sigma_0^2}\right) + \bar{x}\left(\sigma_n^2 \frac{n}{\sigma^2}\right)$

posterior variance. $\sigma_n^2 = \dfrac{1}{\left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)}$

$$\hat{\mu}_n = \frac{\left(\frac{200}{10^2}+\frac{(5)(197.79)}{1}\right)}{\frac{1}{10^2}+\frac{5}{1^2}} = \frac{990.95}{5.01} = 197.79$$

complete the square.

$$\propto \exp\left[-\frac{1}{2}\left(Ax^2 + 2Bx + C\right]\right.$$

$$\propto \exp\left[-\frac{1}{2}\left(Ax^2 + 2Bx\right)\right]. \quad \text{drop } C \propto$$

$$\propto \exp\left[-\frac{1}{2}A\left(x^2 + 2\frac{B}{A}x\right)\right.$$

$$\propto \exp\left[-\frac{1}{2}A\left(x^2 + 2\frac{B}{A}x + D\right)\right]. \quad \text{add } D \propto$$

$$\text{let} \quad D = \left(\frac{B}{A}\right)^2$$

$$\propto \exp\left[-\frac{1}{2}A\left(x - \frac{B}{A}\right)^2\right].$$

$$\propto \exp\left[-\frac{1}{2\sigma^2}\left(x - \mu\right)^2\right].$$

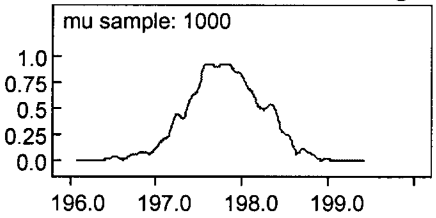
$$\mu = \frac{B}{A}$$

$$\sigma^2 = \frac{1}{A}.$$

Example 8.1.2 Weighing and object

```
# Example 2, Problem 3.2

data

list(X = c(198.14, 198.45, 196.59, 197.64, 198.12), n = 5, sigma = 1)     # frequentist estimate, mean = 197.79

inits

list(mu = 198)

model

model;
{
  for( i in 1 : n ) {
    X[i] ~ dnorm(mu,tau)
  }
  mu ~ dnorm( 200.0,0.01)     # prior mode = 200, 95% prior interval (180,220), Normal(200,10) prec = 1/10 = 0.01
  tau <- 1 / pow(sigma,2)
}
```

Output: posterior density of μ.



node	mean	sd	MC error	2.5%	median	97.5%	start	sample
mu	197.8	0.4371	0.01218	196.9	197.8	198.7	1001	1000

bayesian estimate of μ

bayesian 95% credible interval for μ.

3/15

Example 8.1.3 Counting mice.

prior $\lambda \sim \text{Gamma}(\alpha_0, k_0)$

$$p(\lambda) \propto \lambda^{\alpha_0 - 1} e^{-k_0 \lambda}$$

p 95 $\alpha_0 = 4$, $k_0 = \frac{1}{3}$.

likelihood $x_1, \dots, x_n \overset{iid}{\sim} \text{Poi}(\lambda)$

p. 98 data $n=50$

$t = \sum x_i = 256$.

$$p(x_1, \dots, x_n | \lambda) \propto \prod_{i=1}^{n} \lambda^{x_i} e^{-\lambda}$$

$$= \lambda^{t} e^{-n\lambda} \qquad t = \sum^{n} x_i$$

posterior

$$p(\lambda | x_1, \dots, x_n) \propto \lambda^{\alpha_0 - 1} e^{-k_0 \lambda} \lambda^{t} e^{-n\lambda}$$

$$\propto \lambda^{\alpha_0 + t - 1} e^{-(k_0 + n)\lambda}$$

$\therefore \lambda | x_1, \dots, x_n \sim \text{Gamma}(\alpha_n, k_n)$

$$\alpha_n = \alpha_0 + t$$

$$k_n = k_0 + n.$$

posterior mean. $E[\lambda | x_1, \dots, x_n] = \int_0^\infty \lambda\, p(\lambda | x_1, \dots, x_n)\, d\lambda$

$$= \frac{\alpha_n}{k_n} = \frac{\alpha_0 + t}{k_0 + n}$$

$$= \frac{\alpha_0}{k_0}\left(\frac{k_0}{k_0 + n}\right) + \frac{t}{n}\left(\frac{n}{k_0 + n}\right)$$

p. 98 $\widehat{E[\lambda | x_1, \dots, x_n]} = \frac{4 + 256}{\frac{1}{3} + 50} = 5.17$

Example 8.1.3 Counting mice

Example 3

data

list(T = 256, n = 50)

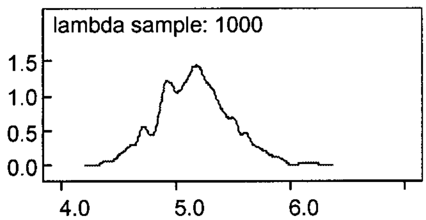
inits

list(lambda = 10)

model

```
model;
{
  T ~ dpois(lambda0)
  lambda ~ dgamma(4,0.3333)
}
```

Output: posterior of λ



node	mean	sd	MC error	2.5%	median	97.5%	start	sample
lambda	5.17	0.3148	0.01088	4.561	5.162	5.825	1001	1000

bayesian estimate of λ

bayesian 95% credible interval for λ

Program 1.1 Power.odc

Model B: Prior and likelihood coding

Model

```
{
        for (i in 1 : N) {
                lambda[i] ~ dgamma(alpha, beta)
                theta[i] <- lambda[i] * E[i]
                y[i] ~ dpois(theta[i])
        }
        alpha ~ dexp(1)
        beta ~ dgamma(0.1, 1.0)
}
```

B: Inits

```
                list(alpha=1,beta=1)
```

A and B: data

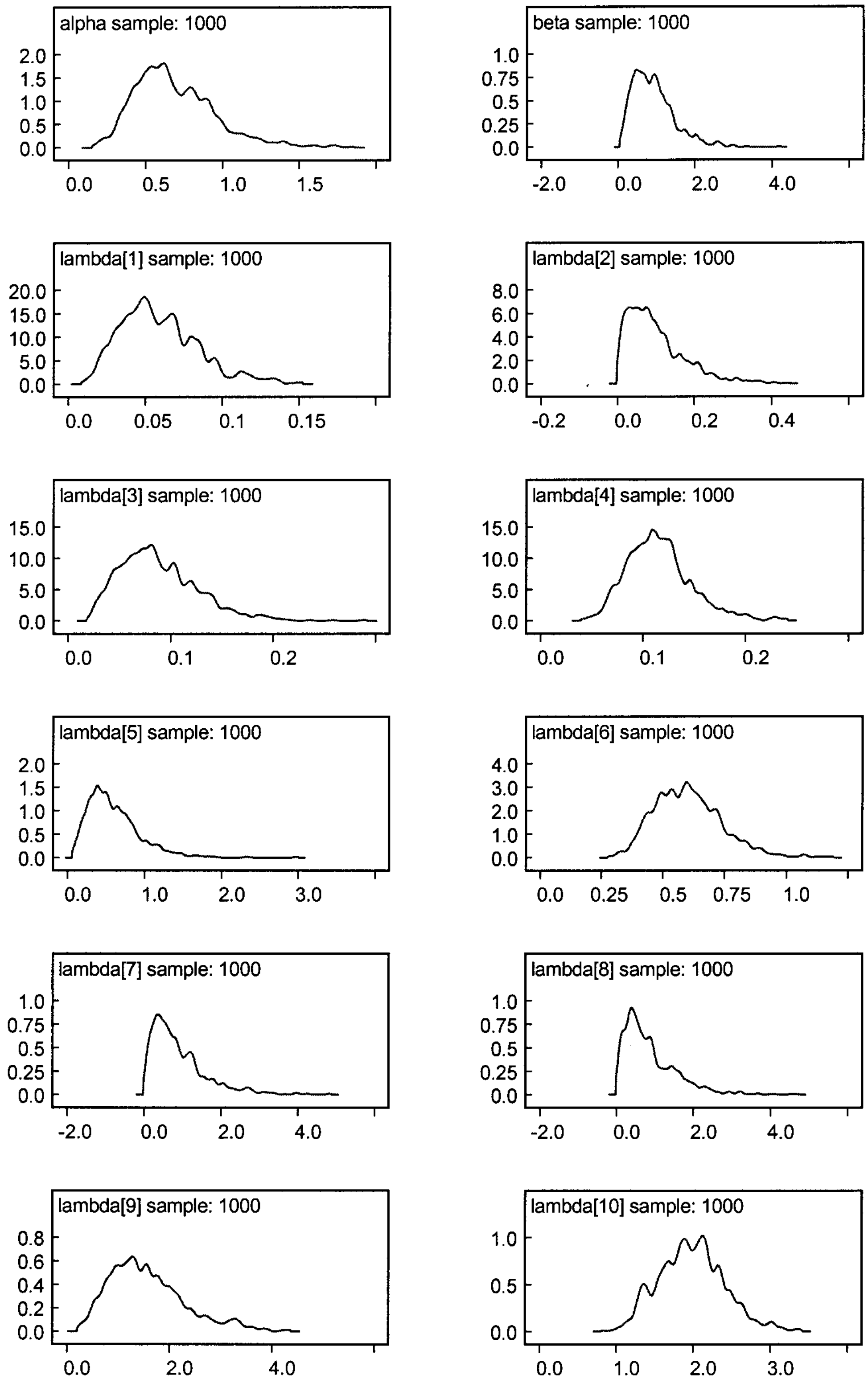
```
list(E = c(94.3, 15.7, 62.9, 126, 5.24, 31.4, 1.05, 1.05, 2.1, 10.5),
      y = c(   5,    1,    5,  14,    3,   19,    1,    1,   4,   22), N = 10)
```

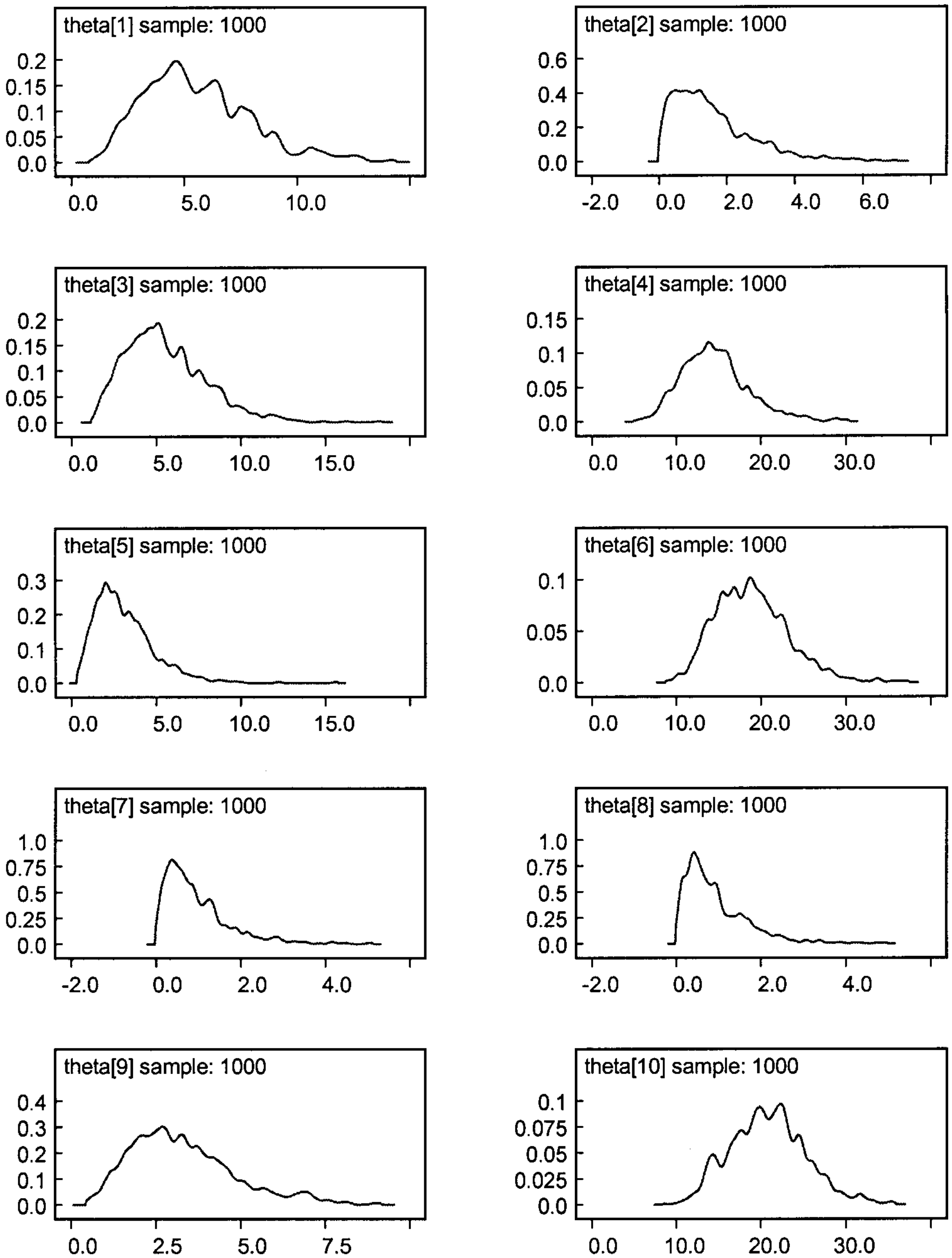
Output:

node	mean	sd	MC error	2.5%	median	97.5%	start	sample
alpha	0.6922	0.2642	0.01444	0.2986	0.6442	1.342	1001	1000
beta	0.9301	0.5498	0.02954	0.1864	0.8518	2.28	1001	1000
lambda[1]	0.05928	0.02521	8.073E-4	0.0206	0.05517	0.1197	1001	1000
lambda[2]	0.09994	0.07535	0.002759	0.007173	0.08192	0.3051	1001	1000
lambda[3]	0.08868	0.03812	0.001307	0.03056	0.08325	0.1804	1001	1000
lambda[4]	0.1161	0.03169	9.349E-4	0.06509	0.1129	0.1904	1001	1000
lambda[5]	0.5974	0.3333	0.01431	0.1458	0.5296	1.398	1001	1000
lambda[6]	0.6026	0.1363	0.004635	0.3825	0.5936	0.9015	1001	1000
lambda[7]	0.9059	0.7322	0.02253	0.08242	0.7118	2.832	1001	1000
lambda[8]	0.8767	0.6935	0.02714	0.07072	0.6983	2.63	1001	1000
lambda[9]	1.578	0.7369	0.02195	0.4871	1.469	3.351	1001	1000
lambda[10]	1.989	0.4236	0.0124	1.249	1.981	2.916	1001	1000
theta[1]	5.59	2.377	0.07613	1.942	5.203	11.29	1001	1000
theta[2]	1.569	1.183	0.04332	0.1126	1.286	4.789	1001	1000
theta[3]	5.578	2.398	0.08219	1.922	5.236	11.35	1001	1000
theta[4]	14.63	3.993	0.1178	8.201	14.22	23.99	1001	1000
theta[5]	3.131	1.746	0.07498	0.7639	2.775	7.324	1001	1000
theta[6]	18.92	4.281	0.1455	12.01	18.64	28.31	1001	1000
theta[7]	0.9512	0.7688	0.02365	0.08654	0.7474	2.973	1001	1000
theta[8]	0.9205	0.7282	0.02849	0.07426	0.7332	2.762	1001	1000
theta[9]	3.314	1.547	0.0461	1.023	3.084	7.038	1001	1000
theta[10]	20.88	4.448	0.1302	13.12	20.8	30.62	1001	1000

posterior estimates

credible intervals


alpha sample: 1000
2.0 1.5 1.0 0.5 0.0
0.0 0.5 1.0 1.5
beta sample: 1000
1.0 0.75 0.5 0.25 0.0
-2.0 0.0 2.0 4.0
lambda[1] sample: 1000
20.0 15.0 10.0 5.0 0.0
0.0 0.05 0.1 0.15
lambda[2] sample: 1000
8.0 6.0 4.0 2.0 0.0
-0.2 0.0 0.2 0.4
lambda[3] sample: 1000
15.0 10.0 5.0 0.0
0.0 0.1 0.2
lambda[4] sample: 1000
15.0 10.0 5.0 0.0
0.0 0.1 0.2
lambda[5] sample: 1000
2.0 1.5 1.0 0.5 0.0
0.0 1.0 2.0 3.0
lambda[6] sample: 1000
4.0 3.0 2.0 1.0 0.0
0.0 0.25 0.5 0.75 1.0
lambda[7] sample: 1000
1.0 0.75 0.5 0.25 0.0
-2.0 0.0 2.0 4.0
lambda[8] sample: 1000
1.0 0.75 0.5 0.25 0.0
-2.0 0.0 2.0 4.0
lambda[9] sample: 1000
0.8 0.6 0.4 0.2 0.0
0.0 2.0 4.0
lambda[10] sample: 1000
1.0 0.5 0.0
0.0 1.0 2.0 3.0


theta[1] sample: 1000
0.0 0.05 0.1 0.15 0.2
0.0 5.0 10.0
theta[2] sample: 1000
0.0 0.2 0.4 0.6
-2.0 0.0 2.0 4.0 6.0
theta[3] sample: 1000
0.0 0.05 0.1 0.15 0.2
0.0 5.0 10.0 15.0
theta[4] sample: 1000
0.0 0.05 0.1 0.15
0.0 10.0 20.0 30.0
theta[5] sample: 1000
0.0 0.1 0.2 0.3
0.0 5.0 10.0 15.0
theta[6] sample: 1000
0.0 0.05 0.1
0.0 10.0 20.0 30.0
theta[7] sample: 1000
0.0 0.25 0.5 0.75 1.0
-2.0 0.0 2.0 4.0
theta[8] sample: 1000
0.0 0.25 0.5 0.75 1.0
-2.0 0.0 2.0 4.0
theta[9] sample: 1000
0.0 0.1 0.2 0.3 0.4
0.0 2.5 5.0 7.5
theta[10] sample: 1000
0.0 0.025 0.05 0.075 0.1
0.0 10.0 20.0 30.0

Program 1.2 Remission data.odc

```
# Prior and Likelihood Model
model {
        for (i in 1:42) {
                t[i] ~ dnorm(mu[Tr[i]], tau[Tr[i]]) I(min[i], )
        }
        for (j in 1:2) {
                tau.mu[j] <- n0*tau[j]
          mu[j] ~ dnorm(0,tau.mu[j])
# independent priors
#       mu[j] ~ dnorm(0,0.001)
      tau[j] ~ dgamma(1,0.001)
      sigma2[j] <- 1/tau[j]
      sigma[j] <- sqrt(sigma2[j])
      }
}

# Data
list(n0=1,t=c(NA,6,6,6,7,NA,NA,10,NA,13,16,NA,NA,NA,22,23,NA,NA,NA,
NA,NA,1,1,2,2,3,4,4,5,5,8,8,8,8,11,11,12,12,15,17,22,23),
min=c(6,0,0,0,0,9,10,0,11,0,0,17,19,20,0,0,25,32,32,34,35,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0),
Tr=c(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,
2,2))

# Inits
list(mu=c(10,10),tau=c(1,1))
```

A survival time is censored if the patient leaves the study before the cancer returns.

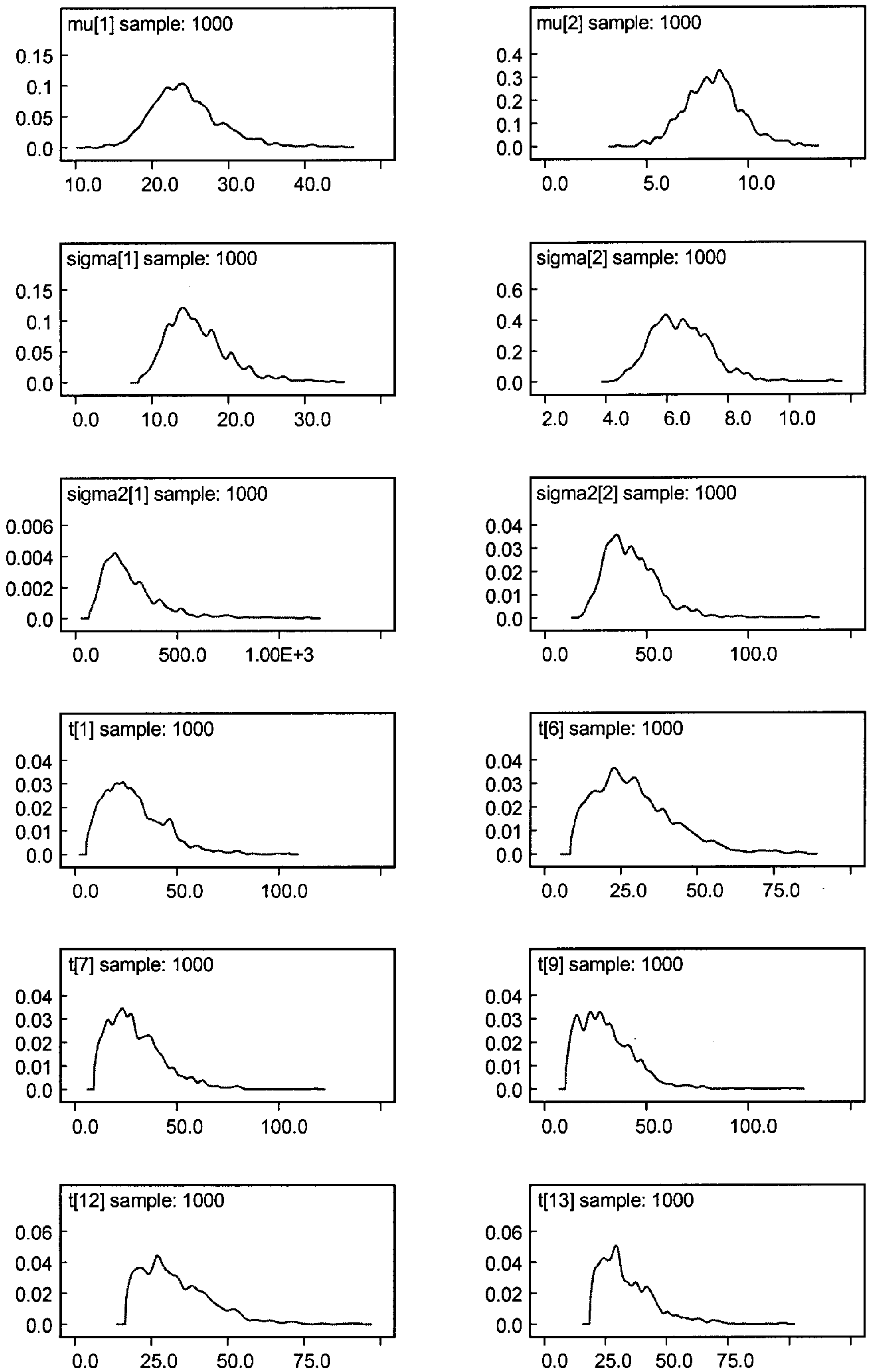
Output:

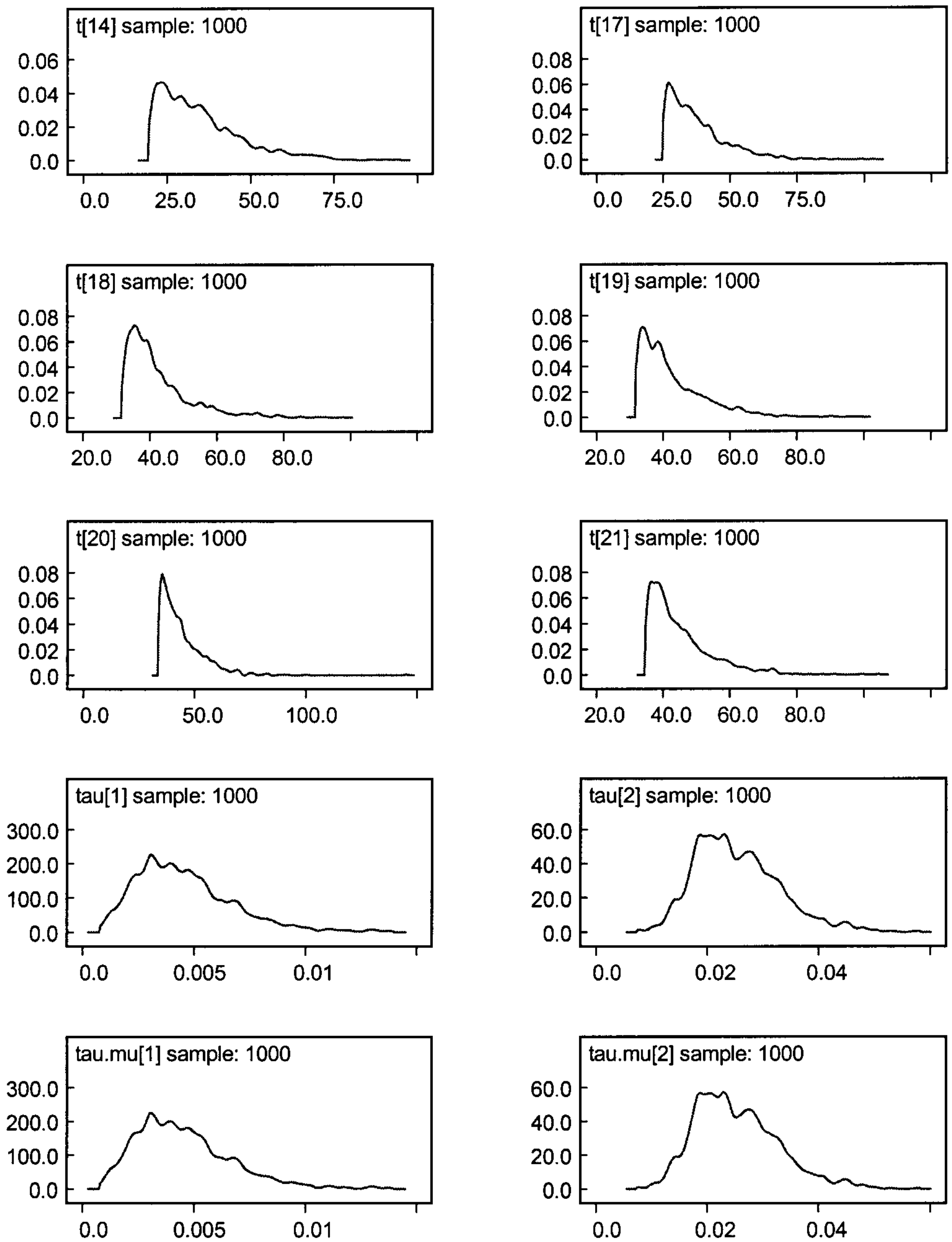
treatment group 1.

node	mean	sd	MC error	2.5%	median	97.5%	start	sample
mu[1]	24.61	4.743	0.3027	17.02	23.98	36.23	1001	1000
mu[2]	8.282	1.388	0.0438	5.564	8.288	11.25	1001	1000
sigma[1]	16.12	4.18	0.273	10.04	15.41	27.05	1001	1000
sigma[2]	6.478	0.9795	0.03307	4.804	6.429	8.628	1001	1000
sigma2[1]	277.3	156.1	10.36	100.7	237.5	732.0	1001	1000
sigma2[2]	42.93	13.46	0.4505	23.08	41.34	74.44	1001	1000
t[1]	28.89	14.9	0.7895	8.026	26.32	65.14	1001	1000
t[6]	29.47	13.1	0.446	10.69	27.4	59.48	1001	1000
t[7]	30.04	13.9	0.6972	11.1	27.7	63.34	1001	1000
t[9]	30.6	13.84	0.5875	12.25	28.3	62.58	1001	1000
t[12]	33.12	12.0	0.5798	17.6	30.6	63.38	1001	1000
t[13]	33.97	11.78	0.4489	19.77	30.83	64.41	1001	1000
t[14]	35.0	12.32	0.4954	20.49	32.28	67.19	1001	1000
t[17]	37.54	11.04	0.444	25.45	34.7	65.17	1001	1000
t[18]	42.47	9.801	0.4519	32.38	39.53	71.21	1001	1000
t[19]	42.75	9.926	0.4332	32.31	39.68	68.39	1001	1000
t[20]	44.09	9.785	0.4516	34.29	41.47	68.46	1001	1000
t[21]	45.16	9.496	0.4357	35.27	42.3	70.68	1001	1000
tau[1]	0.004592	0.002198	1.234E-4	0.001377	0.004216	0.01006	1001	1000
tau[2]	0.02544	0.007504	2.579E-4	0.01344	0.02421	0.04348	1001	1000
tau.mu[1]	0.004592	0.002198	1.234E-4	0.001377	0.004216	0.01006	1001	1000
tau.mu[2]	0.02544	0.007504	2.579E-4	0.01344	0.02421	0.04348	1001	1000

Bayesian estimates

credible intervals.


mu[1] sample: 1000
0.15 0.1 0.05 0.0
10.0 20.0 30.0 40.0
mu[2] sample: 1000
0.4 0.3 0.2 0.1 0.0
0.0 5.0 10.0
sigma[1] sample: 1000
0.15 0.1 0.05 0.0
0.0 10.0 20.0 30.0
sigma[2] sample: 1000
0.6 0.4 0.2 0.0
2.0 4.0 6.0 8.0 10.0
sigma2[1] sample: 1000
0.006 0.004 0.002 0.0
0.0 500.0 1.00E+3
sigma2[2] sample: 1000
0.04 0.03 0.02 0.01 0.0
0.0 50.0 100.0
t[1] sample: 1000
0.04 0.03 0.02 0.01 0.0
0.0 50.0 100.0
t[6] sample: 1000
0.04 0.03 0.02 0.01 0.0
0.0 25.0 50.0 75.0
t[7] sample: 1000
0.04 0.03 0.02 0.01 0.0
0.0 50.0 100.0
t[9] sample: 1000
0.04 0.03 0.02 0.01 0.0
0.0 50.0 100.0
t[12] sample: 1000
0.06 0.04 0.02 0.0
0.0 25.0 50.0 75.0
t[13] sample: 1000
0.06 0.04 0.02 0.0
0.0 25.0 50.0 75.0


t[14] sample: 1000
0.06
0.04
0.02
0.0
0.0 25.0 50.0 75.0
t[17] sample: 1000
0.08
0.06
0.04
0.02
0.0
0.0 25.0 50.0 75.0
t[18] sample: 1000
0.08
0.06
0.04
0.02
0.0
20.0 40.0 60.0 80.0
t[19] sample: 1000
0.08
0.06
0.04
0.02
0.0
20.0 40.0 60.0 80.0
t[20] sample: 1000
0.08
0.06
0.04
0.02
0.0
0.0 50.0 100.0
t[21] sample: 1000
0.08
0.06
0.04
0.02
0.0
20.0 40.0 60.0 80.0
tau[1] sample: 1000
300.0
200.0
100.0
0.0
0.0 0.005 0.01
tau[2] sample: 1000
60.0
40.0
20.0
0.0
0.0 0.02 0.04
tau.mu[1] sample: 1000
300.0
200.0
100.0
0.0
0.0 0.005 0.01
tau.mu[2] sample: 1000
60.0
40.0
20.0
0.0
0.0 0.02 0.04