Variance Stabalizing Transformation

"We already know why we need such a transformation."

Suppose $X_1, \ldots, X_n$ is a sequence of r.v.'s s.t.

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z \sim N(0, \sigma^2). \quad \text{where } \sigma^2 = h(\mu)$$

We seek a transformation $g(\cdot)$ so that $g(\bar{X}_n)$ has a variance which is functionally free of its mean value.

Choose continuous $g(\cdot)$ that has continuous first derivative $g'(\cdot)$

Using the mean value theorem of calculus on $g(\bar{X}_n)$

$$g(\bar{X}_n) - g(\mu) = (\bar{X}_n - \mu)\, g'(\theta_n)$$

$$|\theta_n - \mu| \leq |\bar{X}_n - \mu|.$$

We have

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z \Rightarrow \bar{X}_n \xrightarrow{P} \mu$$

$$\Rightarrow \quad \theta_n \xrightarrow{P} \mu.$$

$$\Rightarrow \quad g'(\theta_n) \xrightarrow{P} g'(\mu)$$ Slutsky.

$$\Rightarrow \quad \frac{g'(\theta_n)}{g'(\mu)} \longrightarrow 1$$

$\therefore$

$$\sqrt{n}\left(g(\bar{X}_n) - g(\mu)\right)$$
$$= \sqrt{n}\,(\bar{X}_n - \mu) \cdot \frac{g'(\theta_n)}{g'(\mu)} \cdot g'(\mu)$$
$$\to Z \cdot g'(\mu).$$

<u>Thm</u>: $$\sqrt{n}\left(g(\bar{X}_n) - g(\mu)\right) \xrightarrow{d} Z \cdot g'(\mu) \sim N\left(0, \{g'(\mu)\}^2 \sigma^2\right)$$

We are considering this because we thought μ and σ^2 are functionally related.

$$\boxed{\sigma^2 = h(\mu)}$$

So we are looking for a transformation $g(\cdot)$ s.t.

$$\{g'(\mu)\}^2 \, h(\mu) = \text{constant}.$$

i.e. the function $g(\cdot)$ is obtained by

$$\boxed{g(\mu) = \int \frac{1}{\sqrt{h(\mu)}} \cdot d\mu.}$$

Hogg and Craig. p. 251 - 252

$$\bar{X}_n \;\dot{\sim}\; N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$g(\bar{X}_n) \;\dot{\sim}\; N\left(g(\mu), [g'(\mu)]^2 \frac{\sigma^2}{n}\right) \quad n \text{ large}$$

or

$$\frac{g(\bar{X}_n) - g(\mu)}{\sqrt{[g'(\mu)]^2 \frac{\sigma^2}{n}}} \;\dot{\sim}\; N(0, 1^2) \quad n \text{ large}$$

Ex 1: Poisson data.

$E[X] = \lambda \qquad Var(X) = \lambda$

$\therefore \quad h(x) = x$

find the transformation.

$$g(x) = \int \frac{1}{\sqrt{h(x)}}\, dx = \int \frac{1}{\sqrt{x}}\, dx = \int x^{-1/2}\, dx$$

$$= 2x^{1/2} = 2\sqrt{x}.$$

So $\quad 2(\sqrt{X_n} - \sqrt{\lambda}) \xrightarrow{d} \frac{Z}{\sqrt{\lambda}} \sim N(0, 1^2)$

Note: $\quad g'(\lambda) = \frac{1}{\sqrt{\lambda}}$

$\{g'(\lambda)\}^2 = \frac{1}{\lambda} \qquad$ So $\{g'(\lambda)\}^2 \cdot \sigma^2 = 1$

Ex 2: χ^2 data $\quad E[X] = n \quad Var(X) = 2n.$

$\therefore \quad h(x) = 2x$

Find the transformation.

$$g(x) = \int \frac{1}{\sqrt{h(x)}}\,dx = \int \frac{1}{\sqrt{2x}}\,dx = \int (2x)^{-1/2}\,dx$$

$$= (2x)^{1/2} = \sqrt{2x}$$

So $\left(\sqrt{2X_n} - \sqrt{2n}\right) \xrightarrow{d} \frac{Z}{\sqrt{2n}} \sim N(0, 1^2).$

Note $g'(n) = \frac{1}{\sqrt{2n}}$

$\{g'(n)\}^2 = \frac{1}{2n} \quad$ So $\{g'(n)\}^2 \sigma^2 = \frac{1}{2n} \cdot 2n = 1$

Ex 3: laboratory situation. $E[X] = \mu$, $Var(X) = k\mu^2$

$\therefore \quad h(x) = kx^2$

find the transformation.

$$g(x) = \int \frac{1}{\sqrt{h(x)}}\, dx = \int \frac{1}{\sqrt{kx^2}}\, dx$$

$$= \frac{1}{\sqrt{k}} \int \frac{1}{x}\, dx = \frac{1}{k} \ln(x)$$

So $$\frac{1}{\sqrt{k}} \left(\ln(X_n) - \ln(\mu) \right) \xrightarrow{d} N \cdot \sqrt{k}\mu$$

$$\sim N(0, 1^2)$$

Note: $g'(\mu) = \frac{1}{\sqrt{k}} \frac{1}{\mu}$

$\{g'(\mu)\}^2 = \frac{1}{k\mu^2}$

So $\{g'(\mu)\}^2 \sigma^2 = \frac{1}{k\mu^2} \; k\mu^2 = 1$

Exercise: Binomial (n, p) data.

Consider $Y_n = \frac{X_n}{n}$.

$$E[Y_n] = p \qquad Var(Y_n) = \frac{p(1-p)}{n}.$$

$$\therefore \quad h(y) = \frac{y(1-y)}{\cancel{x}}$$

we want a transformation

$$g(x) = \int \frac{1}{\sqrt{h(y)}}\, dy = \ldots$$

Exercise: Exponential (θ) $\quad f(x) = \frac{1}{\theta} e^{-x/\theta} \quad x > 0$

$$E[X] = \theta \qquad Var(X) = \theta^2$$

$$\therefore \quad h(y) = y^2$$

we want a transformation

$$g(x) = \int \frac{1}{\sqrt{h(y)}}\, dy = \ldots$$

$$g(x) = \int \frac{\sqrt{n}}{\sqrt{y(1-y)}}\, dy = \sqrt{n} \arcsin \sqrt{y}$$

So $\sqrt{n}\left(\arcsin \sqrt{y_n} - \arcsin \sqrt{p} \right) \xrightarrow{d} Z \frac{\sqrt{n}}{\cancel{p(1-p)}} \sim N(0,1)$

$$g(x) = \int \frac{1}{\sqrt{h(x)}}\, dx = \int \frac{1}{x}\, dx = \cancel{\int x^{-2}\, dx}$$

$$= \cancel{\frac{x^{-1}}{-1} = -\frac{1}{x}} \quad \log x$$

So $\cancel{\left(-\frac{1}{x_n} + \frac{1}{\theta} \right)} \xrightarrow{d} Z/\theta \sim N(0,1)$

$$(\log X + \log \theta) \xrightarrow{d} Z \log \theta \sim N(0,1)$$

$$\{ g'(\theta) \} = \frac{1}{\theta}$$

$$\frac{1}{\theta^2} \cdot \theta^2 = \phi$$