Estimate $I_X(\mu)$

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ assume $\sigma^2 = 1$

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-\mu)^2\right\}.$$

$$= (2\pi)^{-1/2} \exp\left\{-\frac{1}{2}(x-\mu)^2\right\}.$$

$$\log f(x|\mu) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}(x-\mu)^2$$

$$\frac{d}{d\mu}\log f(x|\mu) = (x-\mu)$$

$$L(\mu) = f(x_1, \dots, x_n|\mu) = \prod_{i=1}^{n} f(x_i|\mu)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x_i-\mu)^2\right\}.$$

$$= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\underline{\sum_i (x_i-\mu)^2}\right\}.$$

$$l(\mu) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\underline{\sum (x_i-\mu)^2}$$

$$l'(\mu) = -\sum (x_i-\mu)(-1) = \sum (x_i-\mu) = \sum x_i - n\mu.$$

$$l''(\mu) = -n < 0$$

note: $\sum (x_i-\mu)^2 = \sum (x_i^2 - 2x_i\mu + \mu^2)$

$$= \sum x_i^2 - 2\mu \sum x_i + n\mu^2$$

MLE of μ.

$l'(\mu) = 0$ $\quad\quad \sum x_i - n\mu = 0$

$$n\mu = \sum x_i$$
$$\hat{\mu} = \bar{x}$$

$AV = \frac{1}{n I_X(\mu)}$

$I_{\underset{\sim}{X}}(\mu) = -E[l''(\mu)] = -E[-n] = n$

$AV = \frac{1}{n I_X(\mu)} = \frac{1}{I_{\underset{\sim}{X}}(\mu)} = \frac{1}{n}$

Estimate Fisher's Information.

first estimate $\quad E_\mu[\frac{d}{d\mu} \log f(X|\mu)]$.

$$= \int_{-\infty}^{\infty} \frac{d}{d\mu} \log f(x|\mu) \cdot f(x|\mu)\,dx.$$

$$\approx \frac{1}{n} \sum_{i=1}^{n} \frac{d}{d\mu} \log f(x_i|\hat{\mu})$$

this should be ≈ 0.

second estimate $\quad I_X(\mu) = E[(\frac{d}{d\mu} \log f(X|\mu))^2]$.

$$= \int_{-\infty}^{\infty} (\frac{d}{d\mu} \log f(x|\mu))^2 f(x|\mu)\,dx$$

$$\approx \frac{1}{n} \sum_{i=1}^{n} (\frac{d}{d\mu} \log f(x_i|\hat{\mu}))^2$$

≈ 1

Recall $\quad \frac{d}{d\mu} \log f(x|\mu) = (x-\mu)$

So to produce the estimates using the data set $\underset{\sim}{x} = (x_1, \ldots, x_n)$ and the MLE $\hat{\mu} = \text{mean}(\underset{\sim}{x})$

In R.

$$\text{mean}\left(\frac{d}{d\mu} \log f(\underset{\sim}{x} \mid \hat{\mu}) \right) \quad *$$

should be close to zero. The

$$\text{mean}\left(\left(\frac{d}{d\mu} \log f(\underset{\sim}{x} \mid \hat{\mu}) \right)^2 \right)$$

should be close to $I_X(\hat{\mu})$, and so should

$$\text{var}\left(\frac{d}{d\mu} \log f(\underset{\sim}{x} \mid \hat{\mu}) \right)$$

since the mean (in *) is close to zero.

The $AV = \frac{1}{n}$, so

$$\left(n \cdot \text{mean}\left(\left(\frac{d}{d\mu} \log f(\underset{\sim}{x} \mid \mu) \right)^2 \right) \right)^{-1}$$

and

$$\left(n \cdot \text{var}\left(\frac{d}{d\mu} \log f(\underline{x} \mid \mu) \right) \right)^{-1}$$

should be close.