Define The BVN in matrix notation.

$$(X_1, X_2) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

$$\underset{\sim}{X} \sim BVN(\underset{\sim}{\mu}, \Sigma)$$

$$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \underset{\sim}{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = (2\pi)^{-1} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\underset{\sim}{x} - \underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu})\right\}.$$

$$\Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}.$$

$$I = \Sigma\Sigma^{-1} = \Sigma^{-1}\Sigma$$

$$|\Sigma| = \sigma_1^2\sigma_2^2 - \rho\sigma_1^2\sigma_2^2 = \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$|\Sigma|^{-\frac{1}{2}} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

So

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right\}.$$

The bivariate standard normal

$\underset{\sim}{X} \sim BVN(0, I)$

Apply a linear transformation to $\underset{\sim}{X}$.

$\underset{\sim}{Y} = C\underset{\sim}{X} + \underset{\sim}{\mu}$

$Y_1 = c_{11} X_1 + c_{12} X_2 + \mu_1$
$Y_2 = c_{21} X_1 + c_{22} X_2 + \mu_2$

$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$

$\underset{\sim}{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \quad \underset{\sim}{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

$$E[\underset{\sim}{X}] = \underset{\sim}{0} \qquad \Sigma_X = E[(\underset{\sim}{X} - \underset{\sim}{0})(\underset{\sim}{X} - \underset{\sim}{0})']$$

$$= E[XX'] = E\left[\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} [X_1 \; X_2]\right]$$

$$= E\left[\begin{bmatrix} X_1^2 & X_1X_2 \\ X_1X_2 & X_2^2 \end{bmatrix}\right]$$

$$= \begin{bmatrix} E[X_1^2] & E[X_1X_2] \\ E[X_1X_2] & E[X_2^2] \end{bmatrix}$$

$$= \begin{bmatrix} V(X_1) & Cov(X_1, X_2) \\ Cov(X_1, X_2) & V(X_2) \end{bmatrix}$$

$$= I$$

$$E[\underset{\sim}{Y}] = \underset{\sim}{\mu} \qquad \Sigma = E[(\underset{\sim}{Y}-\underset{\sim}{\mu})(\underset{\sim}{Y}-\underset{\sim}{\mu})']$$

$$= E[C\underset{\sim}{X} \cdot (C\underset{\sim}{X})']$$

$$= E[C\underset{\sim}{X} \cdot \underset{\sim}{X}'C']$$

$$= C\, E[\underset{\sim}{X}\underset{\sim}{X}']\, C'$$

$$= CC'$$

So start with $\underset{\sim}{X} \sim BVN(0, I)$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = (2\pi)^{-1} \exp\left\{-\tfrac{1}{2}(\underset{\sim}{x}-\underset{\sim}{0})' I (\underset{\sim}{x}-\underset{\sim}{0})\right\}.$$

Transform linearly to

$$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = (2\pi)^{-1} |\Sigma|^{-1/2} \exp\left\{-\tfrac{1}{2}(\underset{\sim}{y}-\underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{y}-\underset{\sim}{\mu})\right\}$$

$$\Sigma = CC'$$

$$Y = C X + \mu.$$

$$C X = Y - \mu$$

$$X = C^{-1}(Y - \mu) \qquad \text{so} \quad J = C^{-1}$$

Therefore

$$f_Y(y) = f_X(x)\,|J|$$

$$= f_X(C^{-1}(y-\mu))\,|C^{-1}|$$

$$= (2\pi)^{-1} \exp\left\{-\tfrac{1}{2}\left[C^{-1}(y-\mu)\right]'\left[C^{-1}(y-\mu)\right]\right\} |C^{-1}|$$

$$= (2\pi)^{-1} \exp\left\{-\tfrac{1}{2}\left[(y-\mu)'(C^{-1})'C^{-1}(y-\mu)\right]\right\} |C^{-1}|$$

note: $|C^{-1}| = |C|^{-1}$

$$(C^{-1})'C^{-1} = (C')^{-1}C^{-1} = (CC')^{-1} = \Sigma^{-1}$$

$$= (2\pi)^{-1}|C^{-1}| \exp\left\{-\tfrac{1}{2}(y-\mu)^{-1}\Sigma^{-1}(y-\mu)\right\}.$$

Finally, since $\Sigma = CC'$, $|\Sigma| = |C||C'| = |C|^2$

so $|C| = |\Sigma|^{1/2}$, $|C|^{-1} = |\Sigma|^{-1/2}$, $|C^{-1}| = |\Sigma|^{-1/2}$

$$\therefore\ f_Y(y) = (2\pi)\,|\Sigma|^{-1/2} \exp\left\{-\tfrac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right\}.$$

To simulate general BVN we need to determine C from $\Sigma = CC'$, where

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

Find C from $\Sigma = CC'$. To determine the square root of a matrix, Choleski decomposition.

Here easy.

$$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{bmatrix}.$$

$$= \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} \\ 0 & c_{22} \end{bmatrix}$$

$$cc_{11} = \sigma_1 \qquad 0$$

$$c_{21} = \rho\sigma_2 \qquad c_{22} = \sigma_2\sqrt{1-\rho^2}$$

note $c_{21}^2 + c_{22}^2 = \rho^2\sigma_2^2 + \sigma_2^2(1-\rho^2) = \sigma_2^2$ ✓

$$\underset{\sim}{Y} = C\underset{\sim}{X} + \underset{\sim}{\mu}$$

$$= \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_1 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix} \underset{\sim}{X} + \underset{\sim}{\mu}$$

$$= \begin{bmatrix} \sigma_1 X_1 \\ \rho\sigma_1 X_1 + \sigma_2\sqrt{1-\rho^2}\, X_2 \end{bmatrix} + \underset{\sim}{\mu}$$

$$= \begin{bmatrix} \mu_1 + \sigma_1 X_1 \\ \mu_2 + \sigma_2 \rho X_1 + \sigma_2\sqrt{1-\rho^2}\, X_2 \end{bmatrix}$$

where

$$\underset{\sim}{Y} \sim BVN\,(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$