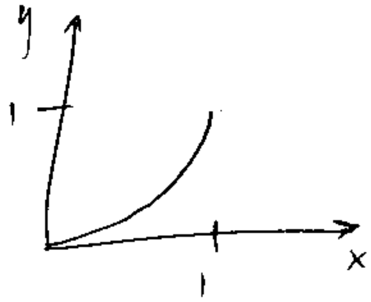
Stat 6401

HW 7 Ch. 6

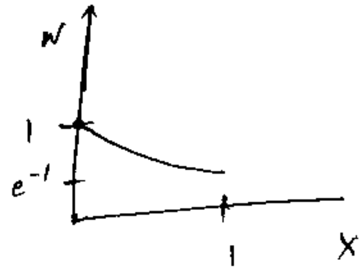
2. $X \sim Unif(0,1)$

a) $Y = X^{1/4}$



$F_Y(y) = P(Y \le y) = P(X^{1/4} \le y)$

$= P(X \le y^4) = F_X(y^4) = y^4 \quad 0 < y < 1$



b.) $W = e^{-X}$

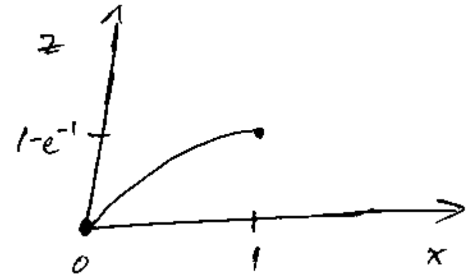
$F_W(w) = P(W \le w) = P(e^{-X} \le w)$

$= P(-X \le \log(w)) = P(X \ge -\log(w))$

$= 1 - P(X \le -\log(w)) = 1 - F_X(-\log(w))$

$= 1 + \log(w) \quad e^{-1} < w < 1$

$f_W(w) = \frac{1}{w} \quad e^{-1} < w < 1$



c.) $Z = 1 - e^{-X}$

$F_Z(z) = P(Z \le z) = P(1 - e^{-X} \le z)$

$= P(1 - z \le e^{-X}) = P(\log(1-z) \le -X)$

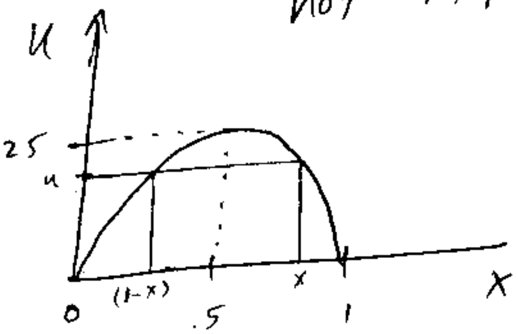
$= P(-\log(1-z) \ge X) = P(X \le -\log(1-z))$

$= F_X(-\log(1-z)) = -\log(1-z) \quad 0 < z < 1 - e^{-1}$

$$f_Z(z) = \frac{1}{1-z} \qquad 0 < z < 1-e^{-1}$$

not 1 to 1 !!!

d) $U = X(1-X)$



$$F_U(u) = P(U \le u) = P(X(1-X) \le u)$$

$$= P\left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-4u} < X \le \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-4u}\right)$$

$x(1-x) = u$

$x - x^2 = u$

$-x^2 + x - u = 0$

$$x = \frac{-1 \pm \sqrt{1-4u}}{-2}$$

$x = \frac{1}{2} + \frac{1}{2}\sqrt{1-4u}$

$x = \frac{1}{2} + \frac{1}{2}\sqrt{1-4u}.$

$$= P\left(X \le \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-4u}\right) - P\left(X \le \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-4u}\right)$$

$$= F_X\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-4u}\right) - F_X\left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-4u}\right)$$

$$= \left[\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-4u}\right] - \left[\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-4u}\right]$$

$$= \tfrac{1}{2}\sqrt{1-4u} + \tfrac{1}{2}\sqrt{1-4u} = \sqrt{1-4u} \qquad 0 \le u \le .25$$

$$f_U(u) = \tfrac{1}{2}(1-4u)^{-1/2}\,|-4| \qquad 0 \le u \le .25$$

$$= 2(1-4u)^{-1/2} \qquad 0 \le u \le .25$$

see p 203
6.3.10

18. $f(x,y) = e^{-y} \qquad 0 < x < y < \infty.$

a) $S = X + Y \qquad Y = S - T$

$T = X \qquad X = T$

$$f_{ST}(s,t) = f_{XY}(x,y) \cdot |J|$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = (0)(-1) - (1)(1)$$

$$= -1$$

$$f_{ST}(s,t) = f_{XY}(t, s-t)\,|-1|$$

$$= e^{-(s-t)} \qquad 0 < 2t < s < \infty$$

b) $f_T(t) = \int_{2t}^{\infty} f(s,t)\, ds = \int_{2t}^{\infty} e^{-(s-t)}\, ds$

$$= \int_{2t}^{\infty} e^{t} e^{-s}\, ds = e^{t} \int_{2t}^{\infty} e^{-s}\, ds$$

$$= e^{t} \left[-e^{-s} \right]_{2t}^{\infty} = e^{t} \left[e^{-2t} \right]$$

$$= e^{-t} \qquad 0 < t < \infty$$

c) $f_S(s) = \int_0^{s/2} f(s,t)\, dt = \int_0^{s/2} e^{-(s-t)}\, dt$

$= \int_0^{s/2} e^{-s} e^{t}\, dt = e^{-s} \int_0^{s/2} e^{t}\, dt$

$= e^{-s} \left[e^{t} \right]_0^{s/2} = e^{-s} \left[e^{s/2} - 1 \right].$

$= e^{-s/2} - e^{-s} \quad 0 < s < \infty.$