Stat 6401
HW 5 Ch. 4

14. $f(x,y) = e^{-(x+y)} \quad 0<x<\infty,\ 0<y<\infty.$

a) $f(x) = \int_0^\infty f(x,y)\,dy = \int_0^\infty e^{-(x+y)}\,dy$

$= e^{-x}\int_0^\infty e^{-y}\,dy = e^{-x} \quad 0<x<\infty.$

$f(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty e^{-(x+y)}\,dx$

$= e^{-y} \quad 0<y<\infty.$

b) $F(x,y) = \int_0^y\int_0^x f(t_1,t_2)\,dt_1\,dt_2$

$= \int_0^y\int_0^x e^{-(t_1+t_2)}\,dt_1\,dt_2$

$= \int_0^y e^{-t_2}\left[\int_0^x e^{-t_1}\,dt_1\right]dt_2$

$= \int_0^y e^{-t_2}\left[-e^{-t_1}\right]_0^x\,dt_2$

$= \int_0^y e^{-t_2}\left[1-e^{-x}\right]dt_2$

$= \left[1-e^{-x}\right]\int_0^y e^{-t_2}\,dt_2$

$= \left[1-e^{-x}\right]\left[-e^{-t_2}\right]_0^y$

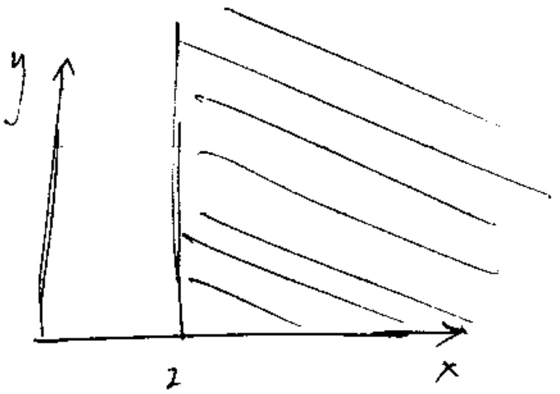
$= \left[1-e^{-x}\right]\left[1-e^{-y}\right] \quad 0<x<\infty,\ 0<y<\infty.$

$= 0 \quad$ otherwise.

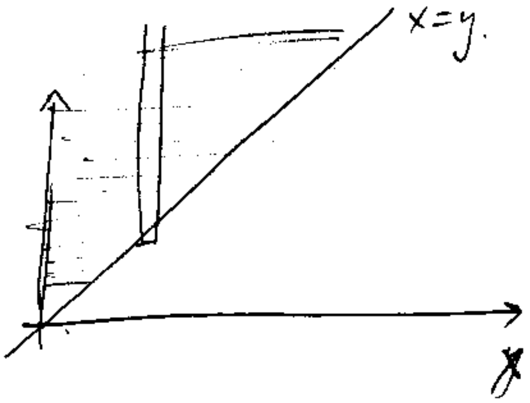
c) $P(X > 2) = \int_2^\infty f(x)\,dx$

$= \int_2^\infty e^{-x}\,dx.$

$= \left[-e^{-x}\right]_2^\infty = e^{-2}$



d) $P(X < Y) = \int_0^\infty \int_x^\infty f(x,y)\,dy\,dx$



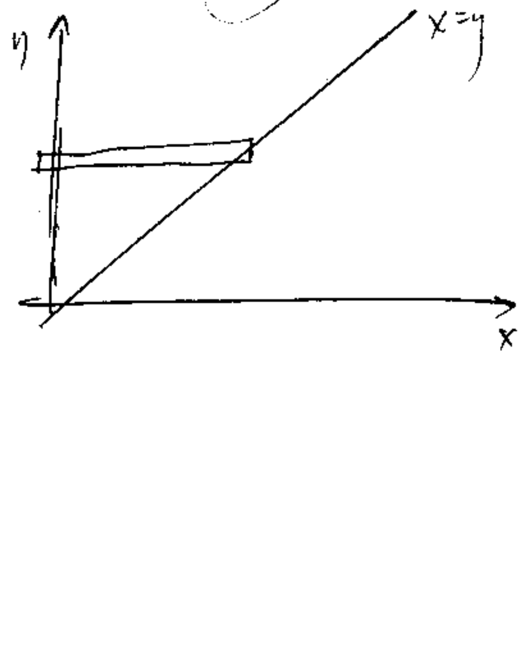
$= \int_0^\infty \int_x^\infty e^{-(x+y)}\,dy\,dx$

$= \int_0^\infty e^{-x}\left[\int_x^\infty e^{-y}\,dy\right]dx.$

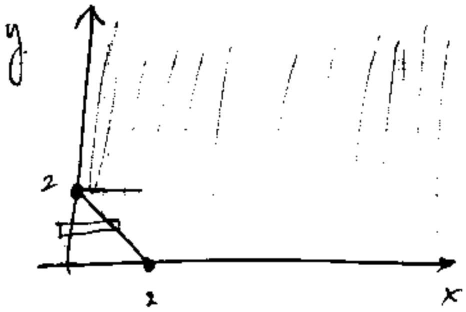
$= \int_0^\infty e^{-x}\left[-e^{-y}\right]_x^\infty dx = \int_0^\infty e^{-x}\left[e^{-x}\right]dx$

$= \int_0^\infty e^{-2x}\,dx = \left[-\frac{1}{2}e^{-2x}\right]_0^\infty = \frac{1}{2}$

or $P(X < Y) = \int_0^\infty \int_0^y f(x,y)\,dx\,dy$



e) $P(X+Y > 2) = 1 - P(X+Y \le 2)$



$= 1 - \int_0^2 \int_0^{2-y} f(x,y)\,dx\,dy.$

$= 1 - \int_0^2 \int_0^{2-y} e^{-(x+y)}\,dx\,dy$

$= 1 - \int_0^2 e^{-y} \left[\int_0^{2-y} e^{-x}\,dx \right] dy.$

$= 1 - \int_0^2 e^{-y} \left[-e^{-x} \right]_0^{2-y} dy.$

$= 1 - \int_0^2 e^{-y} \left[1 - e^{-(2-y)} \right] dy$

$= 1 - \int_0^2 \left[e^{-y} - e^{-2} \right] dy.$

$= 1 - \left[\int_0^2 e^{-y}\,dy - \int_0^2 e^{-2}\,dy \right.$

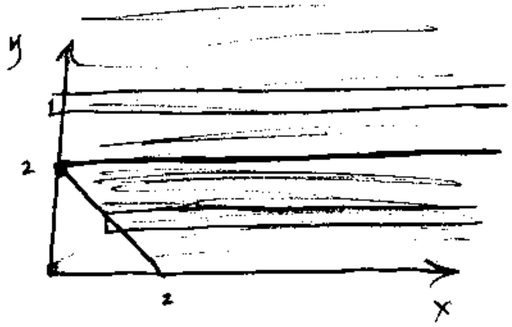
$= 1 - \left[-e^{-y}\Big|_0^2 - e^{-2} y\Big|_0^2 \right].$

$= 1 - \left[1 - e^{-2} - (2e^{-\cancel{2}} - 0) \right].$

$= 1 - \left[1 - e^{-2} - 2e^{-2} \right].$

$= 1 - \left[1 - 3e^{-2} \right]$

$= 3e^{-2}.$



or directly

$$P(X+Y>2) = \iint_{(x,y):\, x+y>2} f(x,y)\, dx\, dy.$$

$$= \int_0^2 \int_{2-y}^{\infty} f(x,y)\, dx.\, dy \; +$$

$$\int_2^{\infty} \int_0^{\infty} f(x,y)\, dx\, dy$$

$$= \dots$$

$$= 3e^{-2}$$

f) Yes. $f(x,y) = f(x) f(y)$ where

$X \sim Exp(1)$ and $Y \sim Exp(1)$

or $F(x,y) = F(x)\, F(y)$.

p. 150 Thm 4.4.1