Modes of Convergence

Let $Z_1, Z_2, ..., Z_n$ be a sequence of jointly distributed random variables defined on the same sample space Ω. Let Z be another random variable defined on this sample space Ω. We now consider what is meant by Z_n "tending to" Z.

Def: (Convergence in probability or weak convergence)

Z_n converges to Z in probability, $Z_n \xrightarrow{p} Z$, if for every $\epsilon > 0$

$$\lim_{n\to\infty} P\left(|Z_n - Z| > \epsilon\right) = 0 \tag{1}$$

Def: (Strong Convergence or almost sure convergence)

Z_n converges to Z almost surely, $Z_n \xrightarrow{a.s.} Z$, if

$$P\left(\lim_{n\to\infty} |Z_n - Z| = 0\right) = 1 \tag{2}$$

Def: (Convergence in Mean Square)

Z_n converges to Z in mean square, $Z_n \xrightarrow{m.s.} Z$, if

$$\lim_{n\to\infty} E[(Z_n - Z)^2] = 0 \tag{3}$$

Thm: (Weak Law of Large Numbers)

Let $X_1, X_2, ..., X_n$ be a sequence of i.i.d. random variables, each having $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, for any $\epsilon > 0$

$$P\left(|\bar{X}_n - \mu| > \epsilon\right) \to 0 \tag{4}$$

as $n \to \infty$.

Proof:

Use Chebychev's Inequality.

So $\bar{X}_n \xrightarrow{p} \mu$.

Thm: (Strong Law of Large Numbers)

Let $X_1, X_2, ..., X_n$ be a sequence of i.i.d. random variables, each having $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$. Then, for any $\epsilon > 0$

$$P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1. \tag{5}$$

So $\bar{X}_n \xrightarrow{a.s.} \mu$.

Def: (Convergence in Distributon)

Z_n converges to Z in distribution, $Z_n \xrightarrow{d} Z$, if,

$$\lim_{n\to\infty} F_{Z_n}(t) = F_Z(t) \tag{6}$$

for all t where $F_Z(t)$ is continuous.

Thm: (Continuity Theorem)

Let $Z_1, Z_2, ..., Z_n$ be a sequence of random variables whose moment generating funtion, $M_{Z_n}(t)$, existis for all $|t| < \epsilon$, for every n. Then

$$F_{Z_n}(z) \to F_Z(z), \tag{7}$$

$Z_n \xrightarrow{d} Z$, if and only if,

$$\lim_{n\to\infty} M_{Z_n}(t) \xrightarrow{d} M_Z(t) \tag{8}$$

where $M_Z(t)$ is the m.g.f. of $F_Z(z)$.

Thm: (Central Limit Theorem)

Let $X_1, X_2, ..., X_n$ be a sequence of i.i.d. random variables, each having mean μ and variance σ^2. The the distribution of

$$\frac{S_n - n\mu}{\sigma\sqrt{n}}, \tag{9}$$

where $S_n = \sum_{i=1}^{n} X_i$, tends to the standard normal distribution

$$\lim_{n\to\infty} P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right) = \Phi(x) \tag{10}$$

where $-\infty < x < \infty$ and $\Phi(x)$ is the c.d.f. of the standard normal distribution.

Thm: Let $F_1, F_2, ...$ be a sequence of c.d.f.'s and the corresponding p.d.f.'s be $f_1, f_2, ...$.

If

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{11}$$

where $f(x)$ is a p.d.f., then F_n converges (in distribution) to the c.d.f. F.

Note: (Stirling Formula)

$$\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\sqrt{\frac{n}{2}}} \to 1 \tag{12}$$

which says $\Gamma(n+1) = n! \approx \sqrt{2\pi}\ n^{n+1/2}\ e^{-n} = \sqrt{2\pi}\ n \left(\frac{n}{e}\right)^n$.