Extra Credit

Direct proof that $\bar{X}$ and S^2 are independent when sampling from the $N(\mu, \sigma^2)$ distribution.

Let X_1, X_2 independent $N(\mu, \sigma^2)$ random variables. (A random sample of size $n = 2$.)

1. Show that $Y_1 = X_1 + X_2$ and $Y_2 = X_2 - X_1$ are independent.
2. What is the distribution of Y_1? What is the distribution of Y_2?
3. Show that $W_1 = \frac{1}{2}Y_1$ and $W_2 = \frac{1}{2}Y_2$ are independent.
   Note:
$$\begin{aligned} W_1 &= \frac{1}{2}Y_1 \\ &= \frac{1}{2}(X_1 + X_2) \\ &= \bar{X} \end{aligned}$$
   and
$$\begin{aligned} W_2 &= \frac{1}{2}Y_2 \\ &= \frac{1}{2}(X_2 - X_1) \\ &= \frac{1}{2}X_2 - \frac{1}{2}X_1 \\ &= X_2 - \frac{1}{2}X_1 - \frac{1}{2}X_2 \\ &= X_2 - \left(\frac{X_1 + X_2}{2}\right) \\ &= X_2 - \bar{X} \end{aligned}$$
   (So $\bar{X}$ and the $n - 1$ deviations from the sample mean are independent.)
4. What is the distribution of W_1? What is the distribution of W_2?
5. Show that since W_1 and W_2 are indenpendent that $W_3 = X_1 - \bar{X}$ is also independent of W_1.
   Note:
$$X_1 - \bar{X} + X_2 - \bar{X} = 0$$
   (So $\bar{X}$ and the first deviation $X_1 - \bar{X}$ are also independent.)
6. Argue that $\bar{X}$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{2}(X_i - \bar{X})^2$ are independent for a random sample of size $n = 2$ from the $N(\mu, \sigma^2)$ distribution.
7. What is the distribution of $\bar{X}$? What is the distribution of S^2?
8. Develop the same results for $n = 3$.
9. Develop the same result for a sample of size n.