

# Practice Problems

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$$3.123 \text{ a) } E[Y] = \left. \frac{d}{dt} m(t) \right|_{t=0} = \left. \frac{1}{6} e^t + \frac{4}{6} 2t + \frac{9}{6} e^{3t} \right|_{t=0} = \frac{7}{3}$$

$$\text{b) } E[Y^2] = \left. \frac{d^2}{dt^2} m(t) \right|_{t=0} = \frac{1}{6} + \frac{8}{6} + \frac{27}{6} = 6$$

$$V(Y) = 6 - \left(\frac{7}{3}\right)^2 = \frac{5}{9}$$

c) Since  $m(t) = E[e^{tY}]$ ,  $Y$  must take only the values  $Y = 1, 2, 3$  with probabilities  $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}$  respectively.

3.135 Let  $Y_1 = \# \text{ def. out of } 5 \text{ from line I.}$

Let  $Y_2 = \# \text{ def. out of } 5 \text{ from line II}$

Then  $Y_1 \sim \text{Bin}(5, p)$  and  $Y_2 \sim \text{Bin}(5, p)$ , where  $p$  is the common probability of def.

Also,  $Y_1 + Y_2 \sim \text{Bin}(10, p)$ .

$$\begin{aligned} P(Y_1 = 2 \mid Y_1 + Y_2 = 4) &= \frac{P(Y_1 = 2 \cap Y_1 + Y_2 = 4)}{P(Y_1 + Y_2 = 4)} \\ &= \frac{P(Y_1 = 2) P(Y_2 = 2)}{P(Y_1 + Y_2 = 4)} = \frac{\binom{5}{2} p^2 q^3 \binom{5}{2} p^2 q^3}{\binom{10}{4} p^4 q^6} \\ &= \frac{\binom{5}{2} \binom{5}{2}}{\binom{10}{4}} = .476 \end{aligned}$$

Note that the end probability does not depend on the value of  $p$ .

3.159 a) Let  $X$  be the number of bacteria colonies in one-cubic-centimeter sample. Then  $X$  has a Poisson distribution with  $\lambda = 2$ . Now let  $Y$  be the number of the four one-cubic-centimeter samples that have one or more bacterial colonies. Note that  $Y \sim \text{Bin}(n=4, p)$ , and

$$p = P(\text{a sample contains one or more colonies}) \\ = P(X \geq 1) = 1 - P(X=0) = 1 - .135 = .865$$

Thus

$$P(\text{at least one sample contains one or more colonies}) \\ = P(Y \geq 1) = 1 - P(Y=0) = 1 - \binom{4}{0} (.865)^0 (.135)^4 = .9997$$

b) We need to find  $n$ , such that

$$P(Y \geq 1) = 1 - P(Y=0) = 1 - \binom{n}{0} (.865)^0 (.135)^n = .95$$

$$\text{or } (.135)^n = .05$$

$$\text{which implies } \log(.135)^n = n \log(.135) = \log(.05)$$

$$\text{so } n = \log(.05) / \log(.135) = 1.496 \approx \textcircled{2} \text{ conservative}$$

$$3.173 \text{ a) } p(10) = \frac{\binom{40}{10} \binom{60}{10}}{\binom{100}{20}} = .119$$

b) binomial approximation with

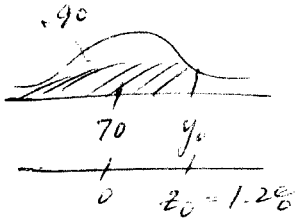
$$p = \frac{r}{N} = \frac{40}{100} = .4 \text{ and } n = 20$$

$$p(10) = \binom{20}{10} (.4)^{10} (.6)^{10} = P(Y \leq 10) - P(Y \leq 9) \\ = .872 - .755 = .117$$

4.128 (optional) a)  $f(y) = \int_{-1}^y \frac{2}{\pi(1+t^2)} dt = \frac{2}{\pi} [\tan^{-1}(t)]_{-1}^y$   
 $= \frac{2}{\pi} [\tan^{-1}(y) - \tan^{-1}(-1)] = \frac{2}{\pi} [\tan^{-1}(y) - (-\frac{\pi}{4})]$   
 $= \begin{cases} \frac{2}{\pi} \tan^{-1}(y) + \frac{1}{2} & -1 \leq y \leq 1 \\ 1 & y > 1 \\ 0 & y < -1 \end{cases}$

b.  $E[Y] = \int_{-1}^1 \frac{2y}{\pi(1+y^2)} dy = \frac{1}{\pi} [\log(1+y^2)]_{-1}^1 = \frac{1}{\pi} (\log 2 - \log 2) = 0$

4.129  $Y \sim N(70, 12^2)$  Find  $y_0$  such that  $P(Y < y_0) = .90$



$$y_0 = \mu + \sigma z_0 = 70 + 12(1.28)$$

$$y_0 = 85.36$$

4.133 a) The variable factor of  $f(y)$  is that of a gamma density with  $\alpha = 2$  and  $\beta = \frac{1}{2}$ . Hence

$$c = \frac{1}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{\Gamma(2)(\frac{1}{2})^2} = \frac{4}{1!} = 4$$

b) Since  $Y \sim \text{Gamma}(2, \frac{1}{2})$

$$E[Y] = \alpha\beta = 1 \quad V(Y) = \alpha\beta^2 = 2(\frac{1}{4}) = \frac{1}{2}$$

c)  $m(t) = \frac{1}{(1-\beta t)^\alpha} = \frac{1}{[1-(\frac{1}{2})t]^2} = (1 - \frac{t}{2})^{-2}$

4.142  $Y =$  time it takes to interview a job applicant

$$f(y) = 2e^{-2y} \quad y > 0$$

The second applicant we have to wait only if the time to interview the first applicant exceeds 15 minutes. Hence

$$P(\text{wait}) = P(Y > \frac{1}{4}) = \int_{\frac{1}{4}}^{\infty} 2e^{-2y} dy = -e^{-2y} \Big|_{\frac{1}{4}}^{\infty} \\ = e^{-\frac{1}{2}} = .61$$

4.152 For  $m = 2$ ,

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$$E(Y) = \int_0^{\infty} \frac{2y^2 e^{-y^2/\alpha}}{\alpha} dy$$

Let  $z = y^2$ . Then  $dz = 2y dy$  and

$$E(Y) = \int_0^{\infty} \frac{\sqrt{z} e^{-z/\alpha}}{\alpha} dz = \int_0^{\infty} \frac{z^{1/2} e^{-z/\alpha}}{\alpha} dz$$

which, when the proper constant is added, will be the integral of the density function of a gamma random variable with parameters  $\frac{3}{2}$  and  $\alpha$ . Then

$$\begin{aligned} E(Y) &= \frac{\alpha^{3/2} \Gamma(\frac{3}{2})}{\alpha} \int_0^{\infty} \frac{z^{1/2} e^{-z/\alpha}}{\Gamma(\frac{3}{2}) \alpha^{3/2}} dz = \frac{\alpha^{3/2} \Gamma(\frac{3}{2})}{\alpha} = \alpha^{1/2} \Gamma(\frac{3}{2}) = \alpha^{1/2} (\frac{1}{2}) \Gamma(\frac{1}{2}) \\ &= \frac{(\alpha\pi)^{1/2}}{2} \end{aligned}$$

where  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$  is shown in exercise 4.162.

Again using the transformation  $z = y^2$ , we find

$$E(Y^2) = \int_0^{\infty} \frac{2y^3 e^{-y^2/\alpha}}{\alpha} dy = \int_0^{\infty} \frac{ze^{z/\alpha}}{\alpha} dz = \frac{\Gamma(2)\alpha^2}{\alpha} \int_0^{\infty} \frac{ze^{-z/\alpha}}{\Gamma(2)\alpha^2} dz = \alpha$$

$$\text{so that } V(Y) = \alpha - [\alpha^{1/2} \Gamma(\frac{3}{2})]^2 = \alpha \left\{ 1 - [\Gamma(\frac{3}{2})]^2 \right\} = \alpha \left[ 1 - \frac{\pi}{4} \right]$$

See Exercise 4.162.

4.153 The density for  $Y$ , the life length of a resistor in thousands of hours, is

$$f(y) = \frac{2ye^{-y^2/10}}{10}, \quad 0 \leq y < \infty$$

$$\begin{aligned} \text{a. } P(Y > 5) &= 1 - P(Y \leq 5) = 1 - \int_0^5 \frac{2ye^{-y^2/10}}{10} dy = 1 - [-e^{-y^2/10}]_0^5 \\ &= 1 - [-e^{-2.5} + 1] = e^{-2.5} \end{aligned}$$

b. Let  $X$  be the number of resistors that burn out prior to 5000 hours. Then  $X$  is a binomial random variable with  $n = 3$  and  $p = 1 - e^{-2.5}$ . Thus,

$$P(X = 1) = \binom{3}{1} (1 - e^{-2.5}) (e^{-2.5})^2 = .0186$$

4.154a When  $m = 1$

$$f(y) = \frac{1}{\alpha} e^{-y/\alpha} \text{ for } 0 \leq y < \infty, \alpha > 0$$

which is the density for an Exponential( $\alpha$ ) random variable.

$$\begin{aligned} \text{b. } E(Y) &= \int_0^{\infty} \frac{m}{\alpha} y^m e^{-y^m/\alpha} dy \\ &= \int_0^{\infty} \frac{1}{\alpha} u^{1/m} e^{-u/\alpha} du \text{ (after making the substitution } u = y^m) \\ &= \alpha^{1/m} \Gamma(1 + 1/m) \end{aligned}$$

$$\begin{aligned} E(Y^2) &= \int_0^{\infty} \frac{m}{\alpha} y^{m+1} e^{-y^m/\alpha} dy \\ &= \int_0^{\infty} \frac{1}{\alpha} u^{2/m} e^{-u/\alpha} du \text{ (after making the substitution } u = y^m) \\ &= \alpha^{2/m} \Gamma(1 + 2/m) \end{aligned}$$

Therefore,

$$V(Y) = \alpha^{2/m} \{ \Gamma(1 + 2/m) - \Gamma^2(1 + 1/m) \}.$$

4.148 (optional) a)  $P(Y \leq 4) = P(X \leq \log 4) = P(Z \leq \frac{\log 4 - 4}{1})$   
 $= P(Z \leq -2.61) = .0045$

b)  $P(Y > 8) = P(X > \log 8) = P(Z > \frac{\log 8 - 4}{1})$   
 $= P(Z > -1.92) = 1 - .0274 = .9726$

4.149 (optional) a)  $E[Y] = e^{u + \sigma^2/2} = e^{3 + 16/2} = e^{11} (10^{-2}g) = 598.74$

$V(Y) = e^{2u + \sigma^2} (e^{\sigma^2} - 1) = e^{6 + 16} (e^{16} - 1) = e^{22} (e^{16} - 1)$   
 $= (e^{38} - e^{22}) (10^{-4}g^2) = 3.1856 \times 10^{12}$

b) with  $k=2$ , consider the interval  $E[Y] \pm 2\sqrt{V(Y)}$

$(598.74 \pm 3,569,637.4) = (-3,569,038.7, 3,570,236.1)$

Since weights are positive, give as the answer  $(0, 3,570,236.1)$

c)  $P(Y < 598.74) = P(\log(Y) < 6.3948)$   
 $= P(Z < \frac{6.3948 - 3}{1}) = P(Z < .8487) = .8023$

4.156 a) Let  $Y$  have an exponential distribution with parameter  $\theta$ . Then  $f(t) = \frac{1}{\theta} e^{-t/\theta}$

and  $1 - F(t) = \int_t^\infty \frac{1}{\theta} e^{-y/\theta} dy = -e^{-y/\theta} \Big|_t^\infty = e^{-t/\theta}$

so  $r(t) = \frac{f(t)}{1 - F(t)} = \frac{1}{\theta}$  which is constant.

b) from 4.152,  $1 - F(t) = \int_t^\infty \frac{my^{m-1} e^{-y/x}}{x} dy$

let  $z = y/x$  so  $dz = my^{m-1} dy$  then

$1 - F(t) = \int_{t/x}^\infty \frac{e^{-z/x}}{x} dz = -e^{-z/x} \Big|_{t/x}^\infty = e^{-t/x}$

so  $r(t) = \frac{m t^{m-1}}{x}$  which is increasing in  $t$  for  $m > 1$ .