Stat 207, Summer 01. Solutions to Midterm

1. (a)

$$E[x_t] = E[\alpha_0 + \alpha_1 t + \alpha_2 t^2 + w_t] = \alpha_0 + \alpha_1 t + \alpha_2 t^2,$$

so $\{x_t\}$ is nonstationary.

(b) Consider ∇^2.

$$\nabla x_t = x_t - x_{t-1} = \alpha_1 + 2\alpha_2 t - \alpha_2 + w_t - w_{t-1},$$

$$\begin{aligned}
\nabla^2 x_t &= \nabla(x_t - x_{t-1}) \\
&= (x_t - x_{t-1}) - (x_{t-1} - x_{t-2}) \\
&= w_t - 2w_{t-1} + w_{t-2} + 2\alpha_2 \\
&= \nabla^2 w_t + 2\alpha_2.
\end{aligned}$$

$$E[\nabla^2 x_t] = E[w_t - 2w_{t-1} + w_{t-2} + 2\alpha_2] = 2\alpha_2 = \mu_x.$$

$$\begin{aligned}
\gamma_{\nabla^2 x_t}(h) &= E[(\nabla^2 x_{t+h} - \mu_x)(\nabla^2 x_t - \mu_x)] \\
&= E[\nabla^2 w_{t+h} \nabla^2 w_t] \\
&= E[(w_{t+h} - 2w_{t+h-1} + w_{t+h-2})(w_t - 2w_{t-1} + w_{t-2})] \\
&= \begin{cases} 6\sigma_w^2, & h = 0, \\ -4\sigma_w^2, & h = \pm 1, \\ \sigma_w^2, & h = \pm 2, \\ 0, & |h| \geq 3. \end{cases}
\end{aligned}$$

Hence $\nabla^2 x_t$ is stationary since the mean and autocovariance do not depend on time t.

2. In the following, to check if it is causal, we want to see if the roots of the AR operator are outside the unit circle.

(a) ARMA(2,0).

$$1 - 1.05B + 0.4B^2 = 0 \quad \Longrightarrow \quad B = 1.3125 \pm 0.62188i.$$

So

$$|B| = \sqrt{1.3125^2 + 0.62188^2} = 1.45 > 1.$$

Since the roots are outside the unit circle, the model is causal.

(b) ARMA(1,0).

$$1 - 1.05B = 0 \quad \Longrightarrow \quad B = 0.9524 < 1,$$

so it is not causal.

(c) ARMA(1,1).

$$1 + 0.8B = 0 \quad \Longrightarrow \quad B = -1.25,\ |B| > 1,$$

so it is causal. In order to check whether the model is invertible, we want to see if the roots of the MA operator are outside the unit circle.

$$1 - 0.25B = 0 \quad \Longrightarrow \quad B = 4,$$

so it is invertible.

3. (a)

$$x_t = x_{t-1} + w_t - \theta w_{t-1},$$
$$(1-B)x_t = (1-\theta B)w_t.$$
$$1 - \theta B = 0 \implies |B| = |\frac{1}{\theta}| > 1,$$

so it is invertible.

$$w_t = \frac{1-B}{1-\theta B}x_t = (1-B)(1+\theta B + \theta^2 B^2 + \cdots)x_t,$$

so

$$\begin{aligned} w_t = (1 \quad &+ \quad \theta B + \theta^2 B^2 + \cdots \\ &- \quad B - \theta B^2 - \cdots)x_t. \end{aligned}$$

Therefore

$$\pi_j = -(1-\theta)\theta^{j-1}.$$

(b)

$$x_t = \sum_{j=1}^{\infty}(1-\theta)\theta^{j-1}x_{t-j} + w_t.$$

(c)

$$E[x_{n+1}|x_n, x_{n-1}, \cdots] = E[\sum_{j=1}^{\infty}(1-\theta)\theta^{j-1}x_{n+1-j} + w_{n+1}|x_n, x_{n-1}, \cdots].$$

$$\begin{aligned} \tilde{x}_{n+1} &= \sum_{j=1}^{\infty}(1-\theta)\theta^{j-1}x_{n+1-j} \\ &= (1-\theta)x_n + \sum_{j=2}^{\infty}(1-\theta)\theta^{j-1}x_{n+1-j} \\ &= (1-\theta)x_n + \theta\sum_{k=1}^{\infty}(1-\theta)\theta^{k-1}x_{n-k} \\ &= (1-\theta)x_n + \theta\tilde{x}_n. \end{aligned}$$

(d)

$$\tilde{x}_{n+1}^{n} = (1-\theta)x_n + \theta\tilde{x}_n^{n-1},$$

the truncated forecasts are computed by letting $\tilde{x}_1^0 = 1$ and the use of the above formula. (see pages 143,144 for EWMA)

4.

(a) MA(2). $x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2}, \quad \theta_1 < 0,\ \theta_2 > 0.$
(b) AR(2). $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t, \quad \phi_1 < 0,\ \phi_2 < 0.$
(c) MA(1). $x_t = w_t + \theta_1 w_{t-1}, \quad \theta_1 > 0.$
(d) AR(1). $x_t = \phi_1 x_{t-1} + w_t, \quad \phi_1 > 0.$