2.3

a

$$
\begin{aligned}
&x_t = 0.8x_{t-1} - 0.15x_{t-2} + w_t - 0.3w_{t-1} \\
&x_t - 0.8x_{t-1} + 0.15x_{t-2} = w_t - 0.3w_{t-1} \\
&(1 - 0.8B + 0.15B^2)x_t = (1 - 0.3B)w_t \\
&(1 - 0.3B)(1 - 0.5B)x_t = (1 - 0.3B)w_t \\
&(1 - 0.3B)x_t = w_t \\
&x_t - 0.5x_{t-1} = w_t \\
&x_t = 0.5x_{t-1} + w_t \qquad ARMA(1,0) \\
&\text{it's causal: } (1 - 0.5B) = 0 \text{ or } B = 2 > 1. \\
&\text{invertible: Yes. } w_t = x_t - 0.5x_{t-1}.
\end{aligned}
$$

b

$$
\begin{aligned}
&x_t = x_{t-1} - 0.5x_{t-2} + w_t - w_{t-1} \\
&x_t - x_{t-1} + 0.5x_{t-2} = w_t - w_{t-1} \\
&(1 - B + 0.5B^2)x_t = (1 - B)w_t \\
&(1 - 0.5B)(1 - 0.5B)x_t = (1 - 0.3B)w_t \qquad ARMA(2,1) \\
&\text{it's causal: } (1 - B - 0.5B^2) = 0 \\
&\text{roots: } 1 \pm i \text{ with module greater than 1} \\
&\text{not invertible:} (1 - B) = 0, \text{ root lies on unit circle}
\end{aligned}
$$

2.6

a

$$
\begin{gathered}
x_t + 1.6x_{t-1} + 0.64x_{t-2} = w_t, \quad \phi_1 = -1.6, \quad \phi_2 = -0.64 \\
(1 + 1.6B + 0.64B^2)x_t = w_t \\
\phi(B)x_t = w_t \\
\psi(B)\phi(B)x_t = \psi(B)w_t \\
(\psi_0 + \psi_1 B + \psi_2 B^2 + \dots)(1 + 1.6B + 0.64B^2) = 1
\end{gathered}
$$

$$
\begin{aligned}
B^0 &: \phi_0 && = 1 \\
B^1 &: \phi_1 + 1.6\phi_0 && = 0 \\
B^j &: \phi_j + 1.6\phi_{j-1} + 0.64\phi_{j-2} && = 0, \quad j \geq 2
\end{aligned}
$$

ACF:

$$\rho(h) - \phi_1\rho(h-1) - \phi_2\rho(h-2) = 0, h = 1, 2, \ldots$$
$$\rho(0) = 1 \quad \text{see example 2, page 103}$$
$$\rho(-1) = \frac{\phi_1}{1-\phi_2}, \quad \rho(1) = \rho(-1)$$

plot:

$$\begin{aligned}
&\rho(0) = 1 \\
&\rho(1) = -\frac{1.6}{1+0.64} = -0.9756 \\
&\rho(2) = -1.6 \times (-0.9756) - 0.64 = -0.9210 \\
&\rho(3) = -1.6 \times 0.9210 - 0.64 \times (-0.9756) = -0.8492 \\
&\rho(4) = -1.6 \times (-0.8492) - 0.64 \times (0.9210) = -0.7095
\end{aligned}$$

b

$$x_t - 0.40x_{t-1} - 0.45x_{t-2} = w_t, \quad \phi_1 = 0.40, \quad \phi_2 = 0.45$$
$$(1 - 0.4B - 0.45B^2)x_t = w_t$$
$$\phi(B)x_t = w_t$$
$$\psi(B)\phi(B)x_t = \psi(B)w_t$$

$$B^j : \phi_j - 0.4\phi_{j-1} - 0.45\phi_{j-2} = 0, \quad j = 0, 1, 2, \ldots$$

ACF:

$$\rho(h) - \phi_1\rho(h-1) - \phi_2\rho(h-2) = 0, h = 1, 2, \ldots$$
$$\rho(0) = 1 \quad \text{see example 2, page 103}$$
$$\rho(-1) = \frac{\phi_1}{1-\phi_2}, \quad \rho(1) = \rho(-1)$$

see plots.

c same idea as $a\&b$.

2.7 ARMA(1,1)

$$x_t = \phi x_{t-1} + \theta w_{t-1} + w_t, \qquad |\phi| < 1$$

from (2.43)

$$r(h) - \phi r(h-1) = 0, \qquad h = 2, 3, \ldots$$

so the general solutions is $r(h) = c\phi^n, \quad h = 1, 2, \ldots$

solve for c using (2.44)

$$\begin{aligned}
r(0) &= \phi r(1) + \sigma_w^2[\theta_0\psi_0 + \theta_1\psi_1] \quad \theta_0 = \psi_0 = 1 \\
r(0) &= \phi r(1) + \sigma_w^2[1 + \theta(\phi + \theta)] \quad \text{see example 2.10}, \psi_1 = \phi + \theta \\
r(0) &= \phi r(1) + \sigma_w^2[1 + \theta\phi + \theta^2] \\
r(0) &= \phi r(1) + \sigma_w^2[1 + \theta\phi + \theta^2] \\
r(1) &= \phi r(0) + \sigma_w^2 \theta_1 \phi_0 \\
r(1) &= \phi r(0) + \sigma_w^2 \theta
\end{aligned}$$

solving for $r(0) \& r(1)$,

$$\begin{aligned}
r(0) &= \sigma_w^2 \frac{1 + 2\theta\phi + \theta^2}{1 - \phi^2} \\
r(1) &= \sigma_w^2 \frac{(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2}
\end{aligned}$$

because $r(1) = cr(0) \Longrightarrow c = \frac{r(1)}{\phi}$ so:

$$r(h) = r(1)\phi^{h-1}$$

or

$$\rho(h) = \frac{(1 + \theta\phi)(\phi + \theta)}{1 + 2\theta + \theta^2}\phi^{h-1}, \quad h \geq 1$$

ARMA(1,1), AR(1)
page 92, $\rho(h) = \phi^h, \quad h \geq 0,$

ARMA(0,1), MR(1)
page 96,

$$\begin{aligned}
\rho(h) &= \frac{\theta}{1 + \theta^2} & h = 1 \\
&= 0 & h > 1
\end{aligned}$$

plots $\phi = 0.6, \theta = 0.9$ see 2.8.

2.12 Cardiovascular mortality:

$$\begin{aligned}
M_t &= 11.45 + 0.43M_{t-1} + 0.44M_{t-2} + w_t \\
Var(w_t) &= 32.5 \\
M_{n+m}^n &= 11.45 + 0.45M_{n+m-1}^n + 0.44M_{n+m-2}^n
\end{aligned}$$

using 2.78

$$P_{n+m}^n = \sigma_w^2 \sum_{j=0}^{m-1} \psi_j^2.$$

2.13

a

$$E[(y-g(x))^2] = E[E[(y-g(x))^2|x]]$$

$$\begin{aligned}
E[(y-g(x))^2|x] &= E[(y-E[y|x]+E[y|x]-g(x))^2|x] \\
&= E[(y-E[y|x])^2|x]E[(E[y|x]-g(x))^2|x] + 2E[(y-E[y|x])(E[y|x]-g(x))|x]
\end{aligned}$$

$E[y|x]-g(x)$ is a constant, so it's easy to show:

$$E[(y-E[y|x])(E[y|x]-g(x))|x] = 0.$$

and as a result:

$$E[(y-g(x))^2|x] \geq E[(y-E[y|x])^2|x]$$

or

$$E[(y-g(x))^2] \geq E[(y-E[y|x])^2]$$

and $E[(y-g(x))^2]$ is minimized when $g(x) = E[y|x]$.

b

$$y = x^2 + z$$

To minimize MSE, set:

$$\begin{aligned}
g(x) &= E[x^2+z|x] \\
&= E[x^2] + E[z|x] \\
&= 1
\end{aligned}$$

$$E(y) = E(x^2) + E(y) = var(x) + 0 = 1$$

$$\begin{aligned}
\text{MSE} &= E[(y-g(x))^2] \\
&= E[(y-1)^2] \\
&= var(y) \\
&= var(x^2+y) \\
&= var(x^2) + var(y) \\
&= 1
\end{aligned}$$

c

$$\begin{aligned}
\text{MSE} &= E[(y-a-bx)^2] \\
&= E[y^2] - 2aE[y] - 2bE[xy] + a^2 + 2abE[x] + b^2E[x^2]
\end{aligned}$$

differentiate by a and b, and set both partial derivatives to zeros, we get:

$$a = E[y] - bE[x] = 1, \qquad b = \frac{E[xy]}{E[x^2]} = 0$$

so $MSE = E[(y-1)^2]$.