Stat 207, Summer 01. Solutions to Homework 2

1. (1.8) (a)

$$E[x_t] = E[\sum_{k=1}^{t} w_k] = \sum_{k=1}^{t} E[w_k] = 0.$$

$$\begin{aligned}
\gamma(s,t) &= E[(x_s - \mu)(x_t - \mu)] \\
&= E[x_s x_t] \\
&= E[\sum_{k=1}^{s} w_k \sum_{l=1}^{t} w_l] \\
&= E[w_1(\sum_{l=1}^{t} w_l) + \cdots + w_s(\sum_{l=1}^{t} w_l)] \\
&= E[w_1^2] + \cdots + E[w_{\min(s,t)}^2] \\
&= \sigma_w^2 \min(s,t),
\end{aligned}$$

since $E[w_m w_n] = 0$ if $m \neq n$.

(b) The series is nonstationary since the autocovariance function

$$\gamma(s,t) = \sigma_w^2 \min(s,t)$$

depends on t. So x_t is not weakly stationary ($\Longrightarrow$ not strictly stationary).

(c) Consider the difference

$$\nabla x_t = x_t - x_{t-1} = \sum_{k=1}^{t} w_k - \sum_{k=1}^{t-1} w_k = w_t.$$

We have

$$E[\nabla x_t] = E[w_t] = 0,$$

and

$$\begin{aligned}
\gamma(h) &= E[(\nabla x_{t+h} - \mu)(\nabla x_t - \mu)] \\
&= E[\nabla x_{t+h} \nabla x_t] \\
&= E[w_{t+h} w_t] \\
&= \begin{cases} \sigma_w^2, & h = 0, \\ 0, & |h| > 0. \end{cases}
\end{aligned}$$

Hence ∇x_t is stationary since its mean is constant and its autocovariance function does not depend on t.

2. (1.9)

$$\begin{aligned}
E[x_t] &= E[u_1 \sin(2\pi\nu_0 t) + u_2 \cos(2\pi\nu_0 t)] \\
&= E[u_1] \sin(2\pi\nu_0 t) + E[u_2] \cos(2\pi\nu_0 t) \\
&= 0.
\end{aligned}$$

In the following, we will use

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B,\\ \cos(A-B) &= \cos A\cos B + \sin A\sin B.\end{aligned}$$

$$\begin{aligned}\gamma(h) &= E[(x_{t+h}-\mu)(x_t-\mu)]\\ &= E[x_{t+h}x_t]\\ &= E[\{u_1\sin(2\pi\nu_0(t+h)) + u_2\cos(2\pi\nu_0(t+h))\}\cdot\{u_1\sin(2\pi\nu_0 t) + u_2\cos(2\pi\nu_0 t)\}]\\ &= E[u_1^2\sin(2\pi\nu_0(t+h))\sin(2\pi\nu_0 t) + u_1u_2\sin(2\pi\nu_0(t+h))\cos(2\pi\nu_0 t)\\ &\quad +u_2u_1\cos(2\pi\nu_0(t+h))\sin(2\pi\nu_0 t) + u_2^2\cos(2\pi\nu_0(t+h))\cos(2\pi\nu_0 t)]\\ &= E[u_1^2\sin(2\pi\nu_0(t+h))\sin(2\pi\nu_0 t) + u_1u_2\sin(2\pi\nu_0(t+h)+2\pi\nu_0 t)\\ &\quad +u_2^2\cos(2\pi\nu_0(t+h))\cos(2\pi\nu_0 t)]\\ &= E[u_1^2]\sin(2\pi\nu_0(t+h))\sin(2\pi\nu_0 t) + E[u_1]E[u_2]\sin(2\pi\nu_0(t+h)+2\pi\nu_0 t)\\ &\quad +E[u_2^2]\cos(2\pi\nu_0(t+h))\cos(2\pi\nu_0 t)\\ &= \sigma^2[\sin(2\pi\nu_0(t+h))\sin(2\pi\nu_0 t) + \cos(2\pi\nu_0(t+h))\cos(2\pi\nu_0 t)]\\ &= \sigma^2\cos(2\pi\nu_0(t+h) - 2\pi\nu_0 t)\\ &= \sigma^2\cos(2\pi\nu_0 h).\end{aligned}$$

So x_t is weakly stationary since its mean is constant and its autocovariance function does not depend on t.

3. (1.14) (a)

$$E[y_t] = E[w_t - \theta w_{t-1} + u_t] = E[w_t] - \theta E[w_{t-1}] + E[u_t] = 0\ (=\mu_y).$$

$$\begin{aligned}\gamma_y(h) &= E[(y_{t+h}-\mu_y)(y_t-\mu_y)]\\ &= E[y_{t+h}y_t]\\ &= E[(w_{t+h}-\theta w_{t+h-1}+u_{t+h})(w_t-\theta w_{t-1}+u_t)]\\ &= E[w_{t+h}w_t - \theta w_{t+h}w_{t-1} + w_{t+h}u_t - \theta w_{t+h-1}w_t + \theta^2 w_{t+h-1}w_{t-1}\\ &\quad -\theta w_{t+h-1}u_t + u_{t+h}w_t - \theta u_{t+h}w_{t-1} + u_{t+h}u_t]\\ &= E[w_{t+h}w_t] - \theta E[w_{t+h}w_{t-1}] - \theta E[w_{t+h-1}w_t] + \theta^2 E[w_{t+h-1}w_{t-1}] + E[u_{t+h}u_t],\end{aligned}$$

since w_t is independent of u_t.
When $h=0$,

$$\gamma_y(0) = E[w_t] + \theta^2 E[w_{t-1}] + E[u_t^2] = (1+\theta^2)\sigma_w^2 + \sigma_u^2;$$

When $h=\pm 1$,

$$\gamma_y(h) = -\theta E[w_t^2] = -\theta\sigma_w^2;$$

When $|h|\geq 2$,

$$\gamma_y(h) = 0.$$

So the ACF

$$\rho_y(h) = \begin{cases} 1, & h=0,\\ -\frac{\theta\sigma_w^2}{(1+\theta^2)\sigma_w^2+\sigma_u^2}, & |h|=1,\\ 0, & |h|\geq 2.\end{cases}$$

(b)

$$E[x_t] = E[w_t] = 0 \ (= \mu_x).$$

$$\begin{aligned}
\gamma_x(h) &= E[(x_{t+h} - \mu_x)(x_t - \mu_x)] \\
&= E[x_{t+h} x_t] \\
&= E[w_{t+h} w_t] \\
&= \begin{cases} \sigma_w^2, & h = 0, \\ 0, & |h| \geq 1. \end{cases}
\end{aligned}$$

$$\begin{aligned}
\gamma_{xy}(h) &= E[(x_{t+h} - \mu_x)(y_t - \mu_y)] \\
&= E[x_{t+h} y_t] \\
&= E[w_{t+h}(w_t - \theta w_{t-1} + u_t)] \\
&= E[w_{t+h} w_t] - \theta E[w_{t+h} w_{t-1}] \quad \text{(since } w_t \text{ and are independent)} \\
&= \begin{cases} \sigma_w^2, & h = 0, \\ -\theta\sigma_w^2, & h = -1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

So the CCF

$$\rho_{xy}(h) = \frac{\gamma_{xy}(h)}{\sqrt{\gamma_x(0)}\sqrt{\gamma_y(0)}} = \begin{cases} \frac{\sigma_w}{\sqrt{(1+\theta^2)\sigma_w^2 + \sigma_u^2}}, & h = 0, \\ -\frac{\theta\sigma_w}{\sqrt{(1+\theta^2)\sigma_w^2 + \sigma_u^2}}, & h = -1, \\ 0, & \text{otherwise.} \end{cases}$$

(c)

$$E[x_t] = 0 \ \text{ and } \ \gamma_x(h) = \begin{cases} \sigma_w^2, & h = 0, \\ 0, & |h| \geq 1, \end{cases}$$

so x_t is stationary.

$$E[y_t] = 0 \ \text{ and } \ \gamma_y(h) = \begin{cases} (1+\theta^2)\sigma_w^2 + \sigma_u^2, & h = 0, \\ -\theta\sigma_w^2, & |h| = 1, \\ 0, & |h| \geq 2, \end{cases}$$

so y_t is stationary.

$$\gamma_{xy}(h) = \begin{cases} \sigma_w^2, & h = 0, \\ -\theta\sigma_w^2, & h = -1, \\ 0, & \text{otherwise.} \end{cases}$$

The series have constant means and the ACF's and CCF only depend on h, so x_t and y_t are jointly stationary.