Solution for stat 207 HW#1

1.3

$$\begin{aligned} r(s,t) &= E[(x_s-\mu_s)(x_t-\mu_t)] \\ &= E[x_s x_t - x_s\mu_t - x_t\mu_s + \mu_t\mu_s] \\ &= E[x_s x_s] - E[x_s]\mu_t - \mu_s E[x_t] + \mu_s\mu_t \\ &= E[x_s x_t] - \mu_s\mu_t \end{aligned}$$

1.7 moving average process.

$$x_t = w_{t-1} + 2w_t + w_{t+1}, \qquad w_t \overset{i.i.d}{\sim} N(0,\sigma_w^2)$$

$$\begin{aligned} E[x_t] &= E[w_{t-1} + 2w_t + w_{t+1}] \\ &= E[w_{t-1}] + 2E[w_t] + E[w_{t+1}] \\ &= 0 \qquad\qquad \text{for all } t \end{aligned}$$

$$\begin{aligned} r(h) &= E[(x_{t+h}-\mu_x)(x_t-\mu_x)] \\ &= E[x_{t+h}x_t] \\ &= E[(w_{t+h-1} + 2w_{t+h} + w_{t+h+1})(w_{t-1} + 2w_t + w_{t+1})] \\ &= E[w_{t+h-1}w_{t-1} + 2w_{t+h-1}w_t + w_{t+h-1}w_{t+1} + 2w_{t+h}w_{t-1} + 4w_{t+h}w_t \\ &\quad + 2w_{t+h}w_{t+1} + w_{t+h-1}w_{t-1} + 2w_{t+h-1}w_t + w_{t+h+1}w_{t+1}] \\ &= E[w_{t+h-1}w_{t-1}] + 2E[w_{t+h-1}w_t] + E[w_{t+h-1}w_{t+1}] + 2E[w_{t+h}w_{t-1}] + 4E[w_{t+h}w_t] \\ &\quad + 2E[w_{t+h}w_{t+1}] + E[w_{t+h-1}w_{t-1}] + 2E[w_{t+h-1}w_t] + E[w_{t+h+1}w_{t+1}] \end{aligned}$$

$$\begin{aligned} h = 0, r(h) &= \sigma_w^2 + 4\sigma_w^2 + \sigma_w^2 = 6\sigma_w^2 \\ |h| = 1, r(h) &= 2\sigma_w^2 + 2\sigma_w^2 = 4\sigma_w^2 \\ |h| = 2, r(h) &= \sigma_w^2 \\ |h| \geq 3, r(h) &= 0 \end{aligned}$$

$$\rho(h) = \frac{r(h)}{r(0)},$$

so

$$\begin{aligned} h = 0, \rho(h) &= 1 \\ |h| = 1, \rho(h) &= \frac{2}{3} \\ |h| = 2, \rho(h) &= \frac{1}{6} \\ |h| \geq 3, \rho(h) &= 0 \end{aligned}$$

1.10

$$x_t = \beta_0 + \beta_1 t + w_t, \quad w_t \overset{i.i.d}{\sim} N(0, \sigma_w^2), \quad \beta_0, \beta_1 \text{ unknown fixed constants}$$

a. nonstationary:

$$\begin{aligned} E[x_t] &= E[\beta_0 + \beta_1 t + w_t] \\ &= \beta_0 + \beta_1 t + E[w_t] \\ &= \beta_0 + \beta_1 t \end{aligned}$$

so x_t is nonstationary since the mean chages with the time.

b.

$$\begin{aligned} \nabla x_t &= x_t - x_{t-1} \\ &= \beta_0 + \beta_1 t + w_t - \beta_0 - \beta_1(t-1) - w_{t-1} \\ &= \beta_1 + w_t - w_{t-1} \end{aligned}$$

stationary:

$$E[\nabla x_t] = E[\beta_1 + w_t - w_{t-1}] = \beta_1 = \mu \quad \forall t$$

$$\begin{aligned} r(h) &= E[(\nabla x_t - \mu)(\nabla x_t - \mu)] \\ &= E[\nabla x_t \nabla x_t] - \mu^2 \\ &= E[(\beta_1 + w_{t+u} - w_{t+h-1})(\beta_1 + w_t - w_{t-1})] - \beta_1^2 \\ &= E[\beta_1^2 + \beta_1 w_t - \beta_1 w_{t-1} + \beta_1 w_{t+u} + w_{t+u} w_t \\ &\quad - w_{t+u} w_{t-1} - \beta_1 w_{t+h-1} - w_{t+h-1} w_t + w_{t+h-1} w_{t-1}] - \beta_1^2 \\ &= E[w_{t+h} w_t] - E[w_{t+h} w_{t-1}] - E[w_{t+h-1} w_t] + E[w_{t+h-1} w_{t-1}] \end{aligned}$$

$$\begin{aligned} h = 0, r(h) &= \sigma_w^2 + \sigma_w^2 = 2\sigma_w^2 \\ |h| = 1, r(h) &= -\sigma_w^2 \\ |h| \geq 2, r(h) &= 0 \end{aligned}$$

so x_t is weakly stationary since its mean is constant and its autocovariance function doesn't depend on the time.